Monte Carlo Integration

$$I = \int_a^b f(x)\,dx$$

Riemann: pick $x_i = hi + a$

$$I = \sum_{i=1}^{n} f(x_i)\,\frac{b-a}{n}$$

This isn't the only way to integrate.

Monte Carlo Integration:

Sample $X \subset U[a,b]$

$$\hat{I} = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)$$

This is technically a random variable.

$$E\{f(x)\} = \int_a^b f(x)\,U[a,b]\,dx = \int_a^b f(x)\frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b f(x_i)\,dx$$

$$E\{\hat{I}\} = \frac{b-a}{n}\sum_{i=1}^{n} E\{f(x)\}$$

$$= \frac{b-a}{n}\sum_{i=1}^{n}\left(\frac{1}{b-a}\int_a^b f(x_i)\,dx\right)$$

$$= \frac{b-a}{n}\,\frac{n}{b-a}\int_a^b f(x)\,dx$$

$$E\{\hat{I}\} = \int_a^b f(x)\,dx$$

Multi-Dimensional Case

$$I = \iiint f(\vec{x})\,d\vec{x}$$

$$\hat{I} = \frac{\forall}{n}\sum_{i=1}^{n} f(\vec{x}_i)$$