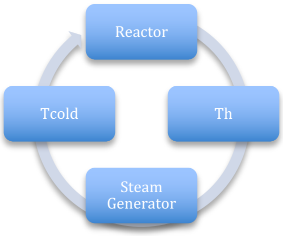

Simplified Steam Plant Transient Model
Revision 15

Demonstrate Temperature Feedback
(Water and Fuel)
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Revised November 3, 2025

1. Reactivity Effects

The reactivity that appears in the kinetics equations is computed from the integration over the reactor volume of the weighted adjoint flux times the result of operating on the initial flux vector by the perturbed differential equation operator. The vectors here are the multigroup energy partitioned adjoint flux and the flux itself. The denominator represents the net production of neutrons per second. This results in a fractional net production divided by the total production.

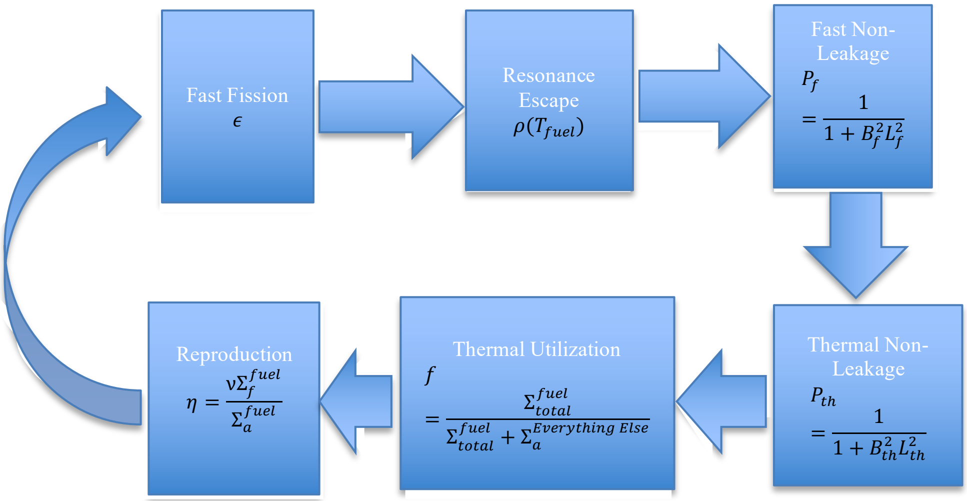
$$\rho = \frac{\left\langle \vec{\phi}_0^{*T}(\vec{r})\,\delta\mathcal{L}(\vec{r},t)\,\vec{\phi}_0(\vec{r}) \right\rangle}{\left\langle \vec{\phi}_0^{*T}(\vec{r})\,\vec{\chi}_t\,\vec{F}^T\,\vec{\phi}(\vec{r},t) \right\rangle}$$

$$\rho = \frac{Neutron\ production\ \ rate - Neutron\,destruction\ \ rate}{Neutron\ production\ \ rate}$$

As discussed in the first paper, we will model the reactivity in terms of six contributing factors associated with the overall neutron life cycle.

Thinking About Reactivity Changes - The Six Factor Formular

$$Keff = \epsilon\rho P_f P_{th} f \eta$$



Fast Fission Factor	ϵ	We will treat the fast fission factor as constant and nearly one. This may not be true in all reactor types.
Resonance Escape Probability	$\rho(T_{fuel})$	As the temperature rises the resonance escape goes down. This is related to a change in the probability that a neutron will be absorbed in the resonances.
Fast Non-Leakage Factor	$P_f = \frac{1}{1+B_f^2L_f^2} \cong e^{-B_f^2L_f^2}$	$B_f^2 = \frac{-\nabla^2\phi_f}{\phi_f}$ Buckling (related to the degree of curvature of the fast flux) L_f^2 Is the square of the fast diffusion length. $L_{fast}^2 = D_{fast}/\Sigma_{total}^{fast}$ (Also known as the slowing down length.)
Thermal Non-Leakage Factor	$P_{th} = \frac{1}{1+B_{th}^2L_{th}^2} \cong e^{-B_{th}^2L_{th}^2}$	$B_{th}^2 = \frac{-\nabla^2\phi_{th}}{\phi_{th}}$Buckling (Curvature of the thermal flux) L_{th}^2 Is the square of the thermal diffusion length. $L_{th}^2 = D_{th}/\Sigma_a^{th}$
Thermal Utilization Factor	$f = \frac{\Sigma_{total}^{fuel}}{\Sigma_{total}^{fuel}+\Sigma_a^{Everything\ Else}}$	The macroscopic cross section is $\Sigma = \sigma \mathrm{N}(\mathrm{Temperature}) \propto \rho_{material}$
Reproduction Factor	$\eta = \frac{\nu\Sigma_f^{fuel}}{\Sigma_a^{fuel}}$	This will change as the composition of the fuel shifts from U^{235} to Pu^{239}. Neutrons created per thermal fuel absorption.

Reactivity in terms of “Worths”

When a sequence of small changes is made to the reactor, the perturbations are considered small enough that we may think of them as differentials. The total reactivity is the sum of these differentials.

$$\rho_{Net} = \rho_{Temperature} + \rho_{Rods} + \rho_{Poison} + \rho_{Fuel\ Burn} + \cdots$$

$$\begin{aligned}\rho_{Net} = \frac{\partial \rho_{Net}}{\partial T} * dT + \frac{\partial \rho_{Net}}{\partial H} * dH + \frac{\partial \rho_{Net}}{\partial Poison} * dPoison \\ + \frac{\partial \rho_{Net}}{\partial Power} * dPower \cdots\end{aligned}$$

Each of the partial derivatives here is given a name in the operational parlance of reactors:

$\frac{\partial \rho_{Net}}{\partial T_{water}}$	α_w Water temperature coefficient
$\frac{\partial \rho_{Net}}{\partial H}$	Differential Rod Worth DRW
$\frac{\partial \rho_{Net}}{\partial Poison}$	Differential Poison Worth
$\frac{\partial \rho_{Net}}{\partial Power}$	α_f Fuel temperature coefficient

The last of these is also often expressed as:

$$\frac{\partial \rho_{Net}}{\partial T_{fuel}} dT_{fuel}$$

This later form is useful in the design process as the designer computes the fuel temperature. The former is more useful to the operator because it allows direct computation based on changes in the overall reactor power.

Buckling and Criticality

If we go back to the neutron diffusion equation, we developed an equation for energy groups with spatial variation within the reactor. Now suppose there were only two groups and suppose that the system is in <u>steady state</u>: the time derivative vanishes, and the delayed neutrons are combined with the prompt neutrons as a constant source. Let the reactor have a uniform composition, so the material properties are all the same over the volume of the reactor.

Ignoring fast fission and thermal group up scattering, we could write the two equations as:

$$0 = \mathrm{D}_{fast}\nabla^2\phi_{fast} - \left(\Sigma_a^{fast} + \Sigma_s{}^{1\to2} \right)\phi_{fast} + \nu\Sigma_f^{th}\phi_{th} + \nu\Sigma_f^{fast}\phi_{fast}$$
$$0 = \mathrm{D}_{th}\nabla^2\phi_{th} - \Sigma_a^{th}\phi_{th} + \Sigma_s{}^{1\to2}\phi_{fast}$$

These equations have a solution where the two fluxes are proportional to each other. In this case the fast and thermal buckling are the same.

With this idea let $B^2 = \frac{-\nabla^2\phi_f}{\phi_f} = \frac{-\nabla^2\phi_{th}}{\phi_{th}}$ and $R = \phi_{th}/\phi_{fast}$

$$0 = -\mathrm{D}_{fast}B^2 - \left(\Sigma_a^{fast} + \Sigma_r^{fast} \right) + \nu\Sigma_f^{th}R \qquad \Sigma_{total}^{fast} = \Sigma_a^{fast} + \Sigma_r^{fast}$$
$$0 = -\mathrm{D}_{th}B^2\phi_{th} - \Sigma_a^{th}\phi_{th} + \Sigma_r^{fast}\phi_{fast}$$

Σ_r^{fast} Is the removal cross section from the fast group.

So

$$L_{fast}^2B^2 + 1 = \nu\Sigma_f^{th}R/\,\Sigma_{total}^{fast} \qquad L_{fast}^2 = D_{fast}/\Sigma_{total}^{fast}$$
$$\phi_{th}/\phi_{fast} = R = \frac{+\Sigma_r^{fast}/\Sigma_a^{th}}{L_{th}^2B^2 + 1} \qquad L_{th}^2 = D_{th}/\Sigma_a^{th}$$

And this gives the criticality condition:

$$(\nu\Sigma_f^{th}/\,\Sigma_{total}^{fast})(\,\Sigma_r^{fast}/\Sigma_a^{th})(\frac{1}{(L_{fast}^2B^2+1)(L_{th}^2B^2+1)}) = 1$$

Or we could write:

$$\boxed{\frac{K_\infty}{(L_{fast}^2B^2+1)(L_{th}^2B^2+1)} = K_\infty P_f P_{th} = 1}$$

We have developed a simplified expression for the conditions for criticality in a homogeneous system. The K_{∞} is a property of the homogeneous medium for a given set of material properties such as temperatures, pressures, and concentrations.

The non-leakage factors are a function of the geometry of the reactor as well as the properties of the medium. It develops that the diffusion constants are approximately: $D_{F,Th} = \frac{1}{3\Sigma_s^{F,Th}}$ hence $L \propto 1/\rho_{water}$ for both the fast and thermal groups. So, for example, if temperature goes up, ρ_{water} goes down so L_{fast} and $L_{thermal}$ go up. Physically this means that loss by leakage will go up.

So, what is the buckling?

$$B^2 = \frac{-\nabla^2 \phi_{th}}{\phi_{th}}$$

This expression gives an indication of the curvature of a scalar flux of multiple variables. If B^2 is positive the flux will be concave. This is a neutron producing region. If the quantity is negative, the flux will be concave, and we will have a neutron absorbing region. In the later case B would be a complex number.

For either the fast or thermal flux we have the following.

$$\nabla^2 \phi + B^2 \phi = 0$$

To understand the implications of this equation, remember that we used Fick's law, $\vec{J} = -D\vec{\nabla}\phi$ and the divergence theorem ($\hat{n}$ is unit outward normal to the surface bounding the volume.), This $\vec{J}$ is the vector flux in some direction, the number of neutrons crossing a unit surface per cm^2 sec.:

$$\iiint \nabla * \vec{J}\, dV = \iint \vec{J} * \hat{n} dS = -\iiint D\nabla^2\phi \, dV$$

Using this we can observe that the buckling within a constant factor, is the fractional leakage of the neutrons in a volume per second:

$$B^2 = \frac{D \iint \vec{J} * \hat{n} dS}{\iiint \phi dV}$$

We will solve $\nabla^2\phi + B^2\phi = 0$ assuming that the flux is near zero on the boundary of our physical reactor. (A better approximation is to extend the outer boundary slightly, but we shall ignore this.)

In Cartesian coordinates

$$\nabla^2\phi + B^2\phi = 0$$

has a solution

$$\phi(x, y, z) = A\cos(B_x x + \delta_x)\cos\left(B_y y + \delta_y\right)\cos(B_z z + \delta_z)$$

Now suppose that we have a rectangular solid reactor with the origin at the center with each dimension extending from $-L_{\{x,y,z\}}/2$ to $L_{\{x,y,z\}}/2$., this would force $\delta_i = 0$ and define all the $B_{x,y,z}$. (n, m, p are integers 1,3,…)

$$\phi(x, y, z) = A * \cos(n\pi x/L_x)\cos\left(m\pi y/L_y\right)\cos(p\pi z/L_z)$$

With this the buckling of the first mode n=m=p=1 is:

$$B^2 = B_x^2 + B_y^2 + B_z^2 = (\frac{\pi}{L_x})^2 + (\frac{\pi}{L_y})^2 + (\frac{\pi}{L_z})^2$$

Higher order modes can exist in some circumstances, but we will not discuss this now. These higher modes do not persist.

If we worked this out for a cylindrical reactor we would get

$$\phi(x, r) = A * \sin\left(\frac{\pi x}{H}\right) \mathrm{J}_0(\frac{2.405r}{R}) \qquad B^2 = (\frac{2.405}{R})^2 + (\frac{\pi}{H})^2$$

J_0 is the zeroth order Bessel function, H is the height of the cylinder, and R is its radius. The origin here is on the axis halfway up.

As noted above, if a region has a weak supply of fission neutrons, the losses dominate the problem, and an external source is required to prevent a zero-flux in steady state. In this case $B^2 < 0$ so B is complex which leads to solutions involving the sinh(x) and cosh(x). This happens if we consider a two-region core, one rodded and one not rodded. As a rule, a region of the core that is neutron producing overall will have a convex flux, regions that are loss dominate will be concave.

How Does the Temperature Reactivity Behave?

Consider first a change in T_{ave}. What happens as temperature goes up in the reactor:

- $Tave \uparrow\ \rho_{H2O} \downarrow L \uparrow Leakage\ \uparrow K_{eff} \downarrow\quad \rho_{Net} \downarrow$ This must dominate
- $Tave \uparrow\ \rho_{H2O} \downarrow \Sigma_a^{H2O} \downarrow K_{eff} \uparrow\ \rho_{Net} \uparrow$
- $Tave \uparrow\ \rho_{H2O} \downarrow \Sigma_a^{Boron} \downarrow K_{eff} \uparrow\ \rho_{Net} \uparrow$

Here we are using two ideas:

Non-Leakage:

$$P_f = \frac{1}{1 + B_f^2 L_f^2}$$

$$P_{th} = \frac{1}{1 + B_{th}^2 L_{th}^2}$$

Thermal Utilization:

$$f = \frac{\Sigma_{total}^{fuel}}{\Sigma_{total}^{fuel} + \Sigma_a^{H2O} + \Sigma_a^{Boron} + \Sigma_a^{Everything\ Else}}$$

So, with temperature the reactivity could go either way. Most designs attempt to have the first effect (leakage) dominate. so

$$\alpha_w = \frac{\partial \rho_{Reactivity\ Net}}{\partial T} < 0\ in\ most\ cases$$

$$\frac{\partial \rho_{Reacitivity\ Net}}{\partial T} = \frac{\partial (1 - \frac{1}{Keff})}{\partial T} = \frac{1}{Keff^2} \frac{\partial Keff}{\partial T}$$
$$= \frac{1}{K} \left(\frac{1}{\rho_{Res}} \frac{\partial \rho_{Res}}{\partial T} + \frac{1}{P_f} \frac{\partial P_f}{\partial T} + \frac{1}{P_{th}} \frac{\partial P_{th}}{\partial T} + \frac{1}{f} \frac{\partial f}{\partial T} \right)$$

We will ignore the $\frac{1}{K}$ term here because it is very close to one.
ρ_{Res} Is the resonance escape probability. Here we have used:

$$Keff = \epsilon \rho P_f P_{th} f \eta$$

Leakage contribution

If we approximate $\frac{1}{1+B_f^2 L_f^2}$ *and* $\frac{1}{1+B_{th}^2 L_{th}^2}$ as $e^{-L_f^2 B^2}$ and $e^{-L_{th}^2 B^2}$ then the leakage terms become $-B^2(2L_f \frac{\partial L_f}{\partial T} + 2L_{th} \frac{\partial L_{th}}{\partial T})$ and the diffusion lengths are inversely proportional to ρ_{water} so these terms are

$$-B^2 \left(2L_f \frac{\partial L_f}{\partial T} + 2L_{th} \frac{\partial L_{th}}{\partial T}\right) = 2B^2 (L_f^2 + L_{th}^2) \frac{1}{\rho_{water}} \frac{\partial \rho_{water}}{\partial T}$$

Thermal Utilization Contribution

$$f = \frac{\Sigma_{total}^{fuel}}{\Sigma_{total}^{fuel} + \Sigma_a^{Everything\ Else}} = \frac{\Sigma_{total}^{fuel}}{\Sigma_{total}^{fuel} + \Sigma_a^{NonWater} + \Sigma_a^{InWater}}$$

Then $\frac{1}{f}\frac{\partial f}{\partial T} = \frac{-\Sigma_a^{InWater}/\rho_{water}}{\Sigma_{total}^{fuel} + \Sigma_a^{NonWater} + \Sigma_a^{InWater}} \frac{\partial \rho_{water}}{\partial T}$

Now we know that $\frac{\partial \rho_{water}}{\partial T} < 0$. Using the absolute value:

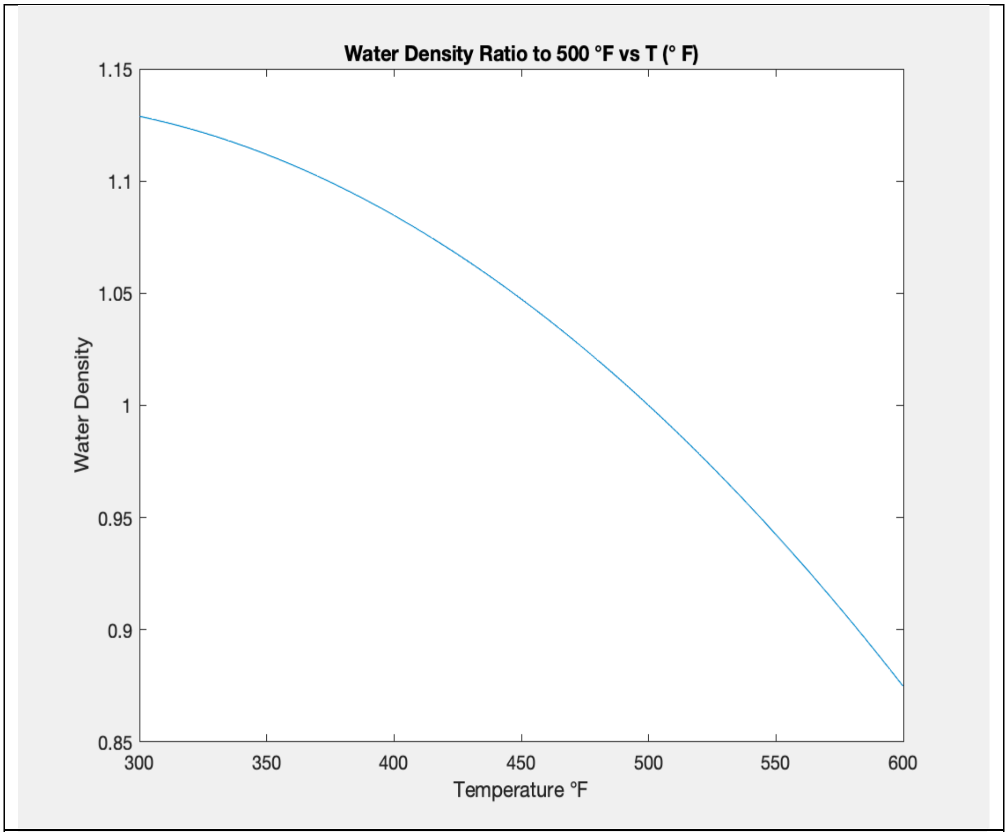
$$\alpha_{water} = \left[-2B^2 (L_f^2 + L_{th}^2) + \frac{1}{(\Sigma_{total}^{fuel} + \Sigma_a^{NonWater})/(\Sigma_a^{Water} + \Sigma_a^{Boron}) + 1}\right] \left|\frac{1}{\rho_{water}} \frac{\partial \rho_{water}}{\partial T}\right|$$

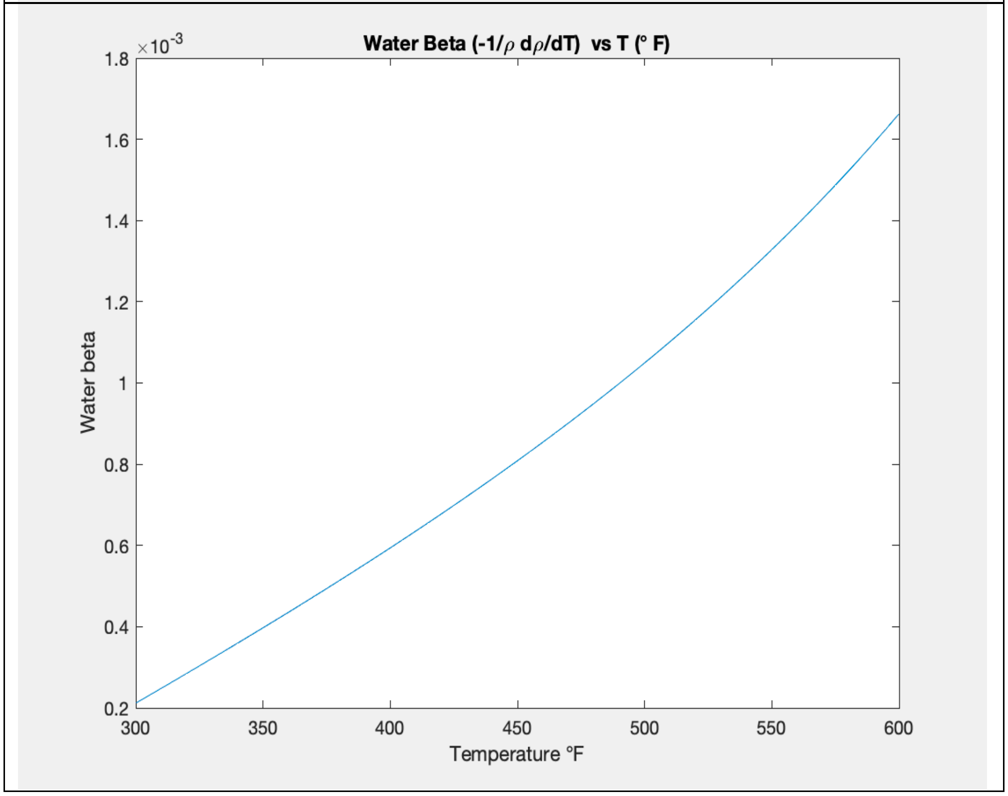
$(L_f^2 + L_{th}^2)$ will be proportional to $\left(\frac{1}{\rho_{water}}\right)^2$ and $\Sigma_a^{Water} + \Sigma_a^{Boron}$ will be proportional to ρ_{water}.

Note that this equation is derived for a homogeneous un-reflected reactor. We will treat $\Sigma_{total}^{fuel} + \Sigma_a^{NonWater}$ as not a function of ρ_{water}. None the less it is instructive in understanding how α_{water} varies. We can see that:

- The magnitude (negative) will become smaller as a reactor becomes larger.
  - A larger reactor design, lower buckling.

  - Rods being pulled out in a bank over core life lowers buckling.
- Increasing the amount of water relative to the other adsorbers will reduce the magnitude if α_{water} was initially negative.
- $\left|\frac{1}{\rho_{water}}\frac{\partial \rho_{water}}{\partial T}\right|$ Increases with water temperature. Also $L_f^2 + L_{th}^2$ increase with water temperature so we expect a more negative α_{water} at operating temperature than we would in a colder reactor.


Water Density Ratio to 500 °F vs T (° F)
Water Density
1.15
1.1
1.05
1
0.95
0.9
0.85
300
350
400
450
500
550
600
Temperature °F


Water Beta (-1/ρ dρ/dT) vs T (° F)
×10^-3
Water beta
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
300
350
400
450
500
550
600
Temperature °F

Define a value $\Psi = (\Sigma_{total}^{fuel} + \Sigma_a^{NonWater})/(\Sigma_a^{Water} + \Sigma_a^{Boron})$. This is the ratio of <u>metal</u> born absorbers to <u>water</u> borne absorbers. Further recognize that the quantity $\beta_{water} = \left|\frac{1}{\rho_{water}}\frac{\partial \rho_{water}}{\partial T}\right|$ is a tabulated value. Now because the diffusion lengths depend inversely on the density of water, we will establish those as $L_f = L_{f0}\rho_0/\rho, \; and \; L_{th} = L_{th0}\rho_0/\rho$. Also, the water born cross sections will depend directly on the water density so:

$$\Psi = \Psi_0\rho_0/\rho$$

$$\alpha_{water} = \left[-2B^2(L_{f0}^2 + L_{th0}^2)\rho_0/\rho + \frac{1}{\Psi_0\rho_0/\rho + 1}\right]\beta_{water}$$

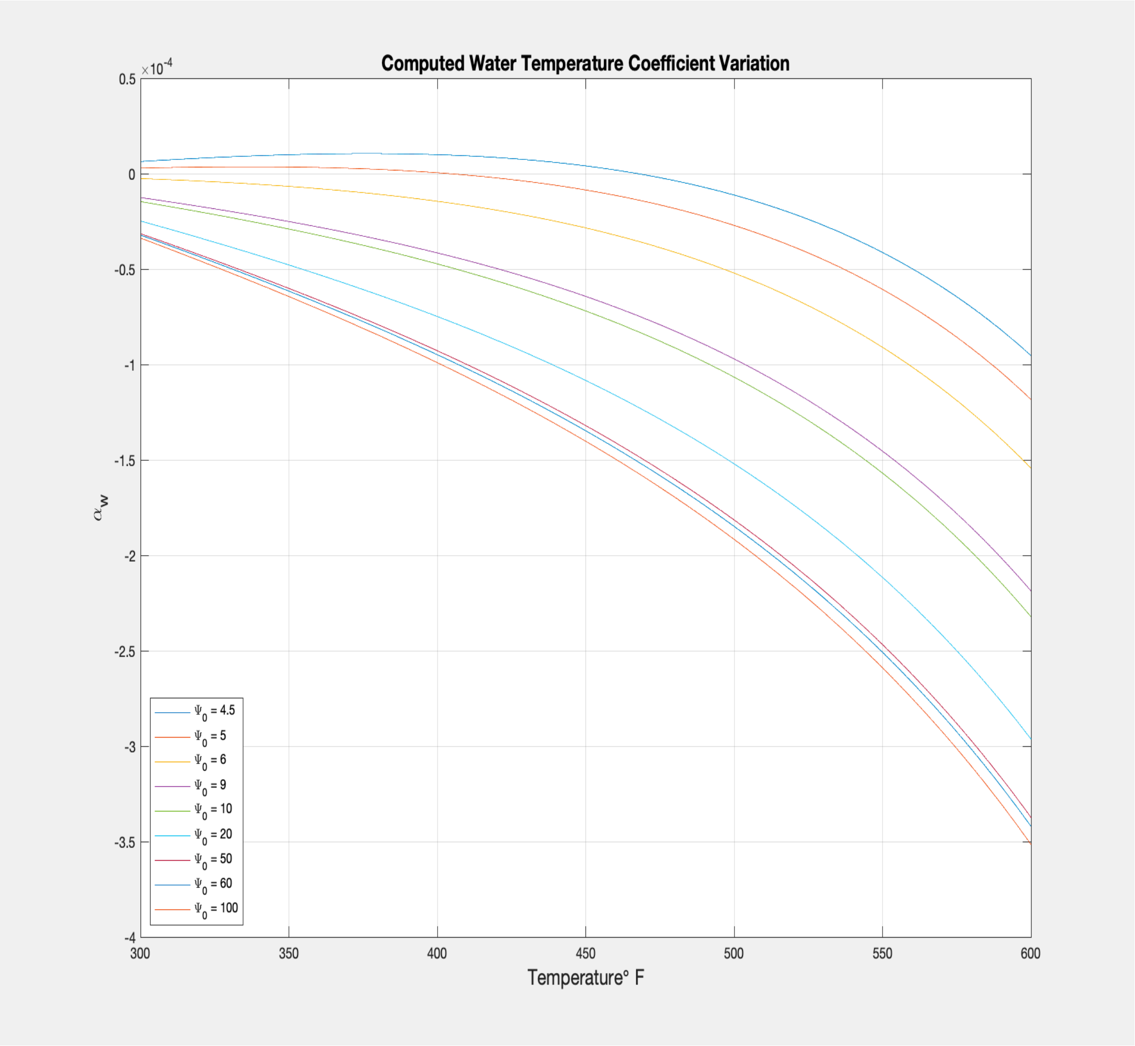
Assume the following purely hypothetical numbers:

L_{f0}	6 cm (500°F)
L_{th0}	4 cm (500°F)
H	130 cm
R	80 cm
T_0	500 °F

This yields a buckling of

$$B^2 = (\frac{2.405}{R})^2 + (\frac{\pi}{H})^2 = 0.0015.$$

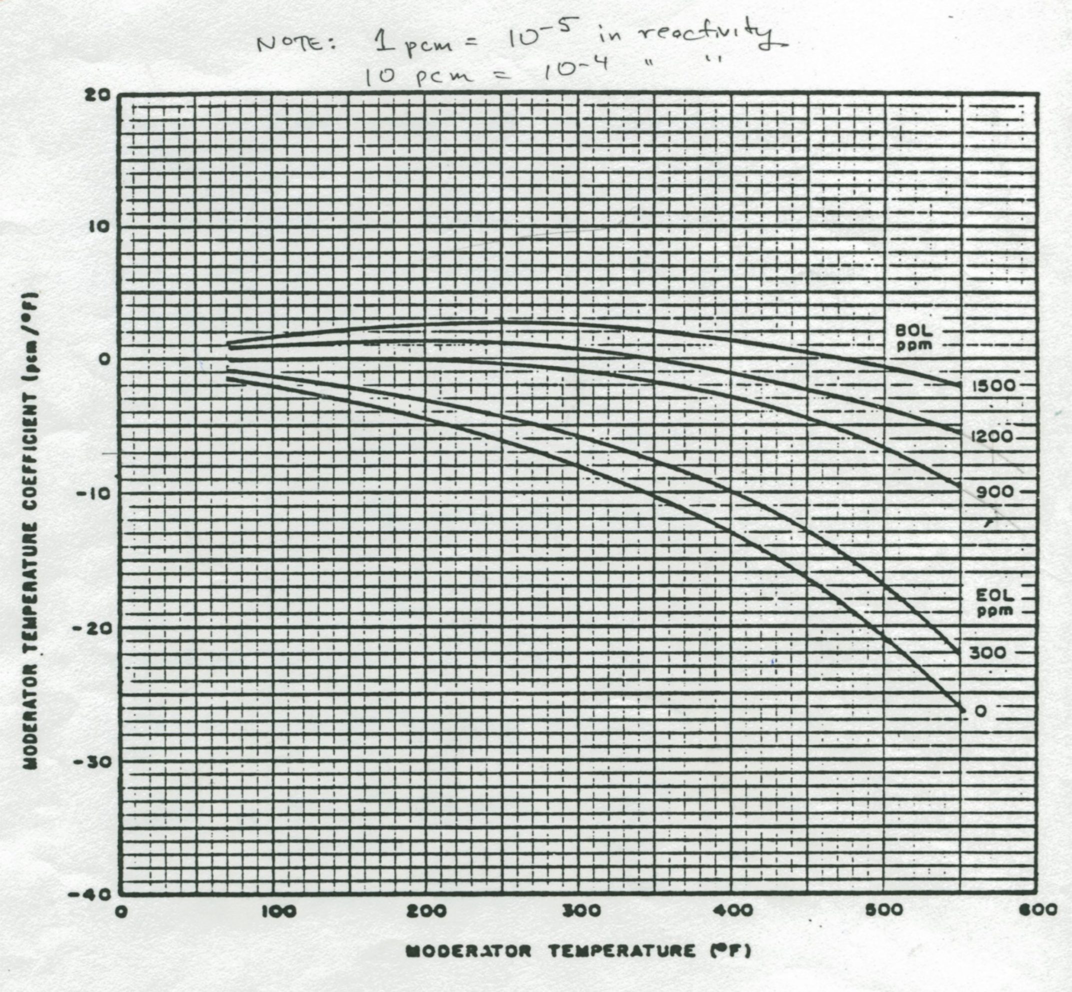
Putting it all together we get the following example:



Psi is proportional to the metal to water ratio in the system. As Psi goes up the temperature coefficient tends towards the negative.

The discussion provided above represents an un-reflected uniform core. Real cores are not uniform and present a much more complex situation. In some cases, neutron poisons are installed as pins or control rods. To a degree these act as internal leakage surfaces and help to make α_{water} negative. Below observe an actual α_{water} plot for an operating reactor design. We can see the impact of temperature and boron in these plots.

Moderator Temperature Coefficient Westinghouse SNUPS Plant

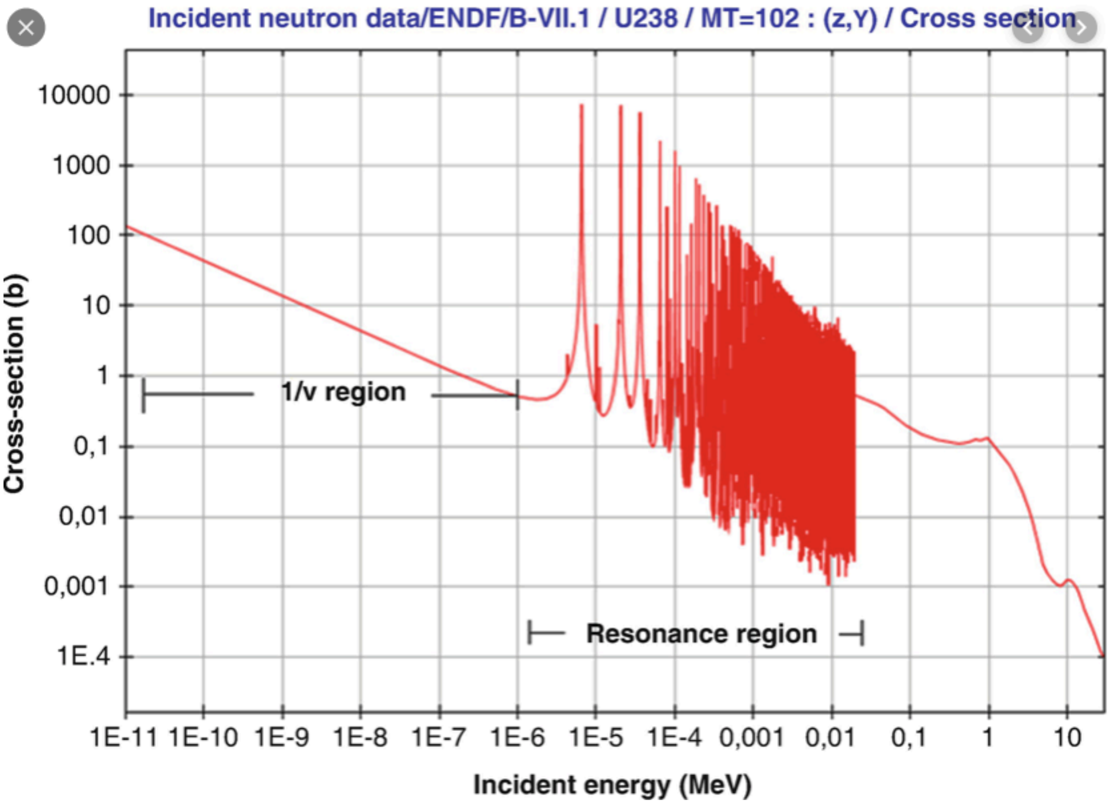


BOL is Beginning of life
EOL is end of life
1pcm is 1.0e-5

For cores that maintain all rods out and control with boron concentration, as the core ages fuel reactivity is reduced. The amount of boron needed is reduced so the positive contribution to α_w is reduced. Remember these reactors are run with rods fully out most of the time.

Reactivity related to Fuel Temperature

(Resonance Escape - Reactor Power)



Many isotopes have cross sections that vary significantly with neutron energy. As shown here for U-238, there is a region from about 1ev to about 10kev with high neutron resonances. The atoms exist within the fuel matrix and the matrix temperature is related to the vibrational energy of the atoms. In low enrichment cores this has a major impact on the resonance escape probability and hence on reactivity.

- The atoms each have resonant peaks
- The atoms actually vibrate randomly with a distribution of energies given by statistical mechanics.
- As such a given neutron is presented an energy window for capture in a specific resonance that is larger than the width of the resonance in the reference frame of the atom.

In computing the resonant absorption probability and the resonant escape probability the product of the energy dependent flux is multiplied by the cross section for a resonance and integrated over the energy. The broadening of the peak in this product results in larger adsorption at higher temperature. In the case of this neutron adsorption, this is known as doppler broadening.

The fuel temperature rises as core power rises. The Power Coefficient of reactivity, α_F is expressed in one of two ways depending upon its application. If the power level is being used to specify the resultant reactivity, α_F will have units related to pcm/power. In this use the assumption is made that the temperatures in the fuel and related structure reach equilibrium in zero time.

$$\Delta\rho_{fuel} = \alpha_F(P - P_0)$$

Alternatively, the Power Coefficient of reactivity, α_F, may be expressed in pcm/temperature. This form of the coefficient is used for computing transient behavior in fuel.

$$\Delta\rho_{fuel} = \alpha_F\delta T_{fuel}$$

Computing Reactivity and Power in the Presence of α_F

Consider a prompt jump-based computation. Here we are assuming the left-hand side of the first reactor kinetics equation is zero and basing the transient calculation on an initial discontinuity in power followed by a solution to the precursor differential equations. This then uses an algebraic power calculation. We will use the power-based reactivity calculation, $\Delta\rho_{fuel} = \alpha_F(P - P_0)$.

We have:

$$P = \frac{\Lambda \sum_i \lambda_i C_i + \Lambda S}{\beta - \Delta\rho_{fuel} - \rho_{other}}$$

So, we need to solve these two equations for P and $\Delta\rho_{fuel}$.

Let c = $\frac{\Lambda \sum_i \lambda_i C_i(t) + \Lambda S}{\alpha_F}$ and $b = \frac{(-\beta + \rho^{other} - \alpha_f P_0)}{\alpha_F}$ with these definitions we have a quadratic equation in power.

$$P(t)^2 + bP(t) + c = 0$$

So

$$P(t) = \left(-b + \sqrt{b^2 - 4c}\right)/2$$

The negative radical is rejected because power is positive, and b must be positive to meet the restrictions of the prompt jump assumption.

Finally:

$$\Delta\rho_{fuel} = \alpha_F(P(t) - P_0)$$

Reactivity Examples

Consider the following data:

- The Fuel Reactivity power coefficient is: $\alpha_F = -10$ pcm/(% power).
- The Temperature coefficient is: $\alpha_T = -20$ pcm/ °F
- The initial T_{ave} is taken as 500°F.
- Rod Reactivity +40 pcm/inch out. One Step is 5/8 inch.

a. Suppose a transient happens in which power is increased by 25%, and Rods are pulled 8 steps out. What will happen to plant temperature in the final steady state?

The temperature reactivity needs to balance the net reactivity inserted.

Needed temperature reactivity to cancel reactivity change =
-[(-10 pcm/% * 25%) +40 pcm/in * 5/8 in/step *8 steps)]= -50 pcm

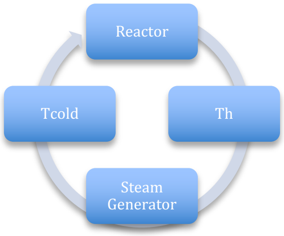
Needed temperature reactivity = -50
ΔT = -50 pcm/(-20 pcm/ °F) = 2.5°F

b. A plant has a steady startup rate of 1 dpm below the point of adding heat. How much reactivity is present? How much reactivity will remain when power turns if the heat up rate is 0.092 °F/sec at its peak?
Assume: $\beta = 640\, pcm \text{ and } \lambda_{eff} = \frac{0.1}{sec}$. Ignore sources and changes in λ_{eff}.

$SUR = 26.06\, dpm - sec \left\{ \frac{\dot{\rho} + \lambda_{eff}\rho}{\beta - \rho} \right\}$	$\rho = \frac{\frac{SUR\, \beta}{(26.06 dpm - sec)} - \dot{\rho}}{\lambda_{eff} + \frac{SUR}{(26.06 dpm - sec)}}$
a) $\dot{\rho} = 0,\ SUR = 1\, dpm,\ \lambda_{eff} = 0.1/sec$	$\rho = 177.48\, pcm$
b) $\dot{\rho} = \alpha_T * 0.092° \frac{\mathrm{F}}{\mathrm{sec}} = -1.84\, \frac{pcm}{sec}$ c) $\mathrm{SUR} = 0.0\ \mathrm{dpm}$	$\rho = 18.46\, pcm$

Keep in mind that in each case where we are using the one delayed group approximation our numbers should be considered suspect at best. The failure to account for $\dot{\lambda}_{eff}/\lambda_{eff}$ leads to significant error.

2. Interaction of Water Volumes and Temperature



Each water volume has a heat capacity. We combine the reactor volume with the hot leg and the steam generator volume with the cold leg such that there is a heat capacity for each, C_r and a C_s. Heat flows into the system from the reactor and out of the volume from the steam generator.

The kinetics Equations will be the same as they were in the low power (no feedback) operations section of the course (with reactor power less than the point of adding heat).

When power is high enough to impact the temperature, this will impact the temperature reactivity. The temperature of the system is a function of the steam demand history and power history. The difference between the power and the steam demand is the “mismatch”.

If (Power – Steam Demand > 0) Heat is added to the coolant and temperature goes up.

If (Power – Steam Demand < 0) Heat is removed from the coolant and temperature goes down.

In this case we are looking at this as two water masses, reactor and steam generator ignoring the pipe transport times. Hot side water from the reactor flows into the steam generator and water from the cold side of the steam generator flows into the reactor. So, for example water coming from the reactor would heat the steam generator and heat removed by steam would cool the steam generator. The water is force along the system by a pump and we will ignore any heat introduced by the pump.

The rates of change of the hot and cold leg temperatures are proportional to the difference between the water entering each volume and the water leaving the volume. If this difference is positive the temperature will rise (positive derivative) otherwise it will fall (negative derivative). For the reactor the water leaving is T_h and the water entering is T_c. The reverse is true for the steam generator. The heat introduced to the reactor when divided by the heat capacity of the water in the reactor is a positive contribution to the rate of change of temperature. The heat removed from the steam generator divided by its heat capacity contributes a negative contribution to the steam generator.

Each volume is experiencing a “fill and drain” transient. If we held the temperature of the incoming water constant and ignored any other heat introduction, the temperature of that volume would be a transient from its initial temperature to the temperature of the entering water with an exponential transient:

$T_h(t) = T_h^{initial} e^{-t/\tau_r} + T_c(1 - e^{-t/\tau_r})$	$\tau_r = \frac{Reactor\ Water\ Volume}{Reactor\ Water\ Flow\ Rate}$

We will ignore the change in water density that this change in temperature creates. So, we will directly relate system volume with water mass.

$$\frac{dT_h}{dt} = \frac{T_c - T_h}{\tau_r} + \frac{P_r}{C_r}$$

τ_r and C_r apply to the source including the hot side piping. Both quantities are proportional to the mass of the water that is contained in the volume. (We will ignore the effects of the metal for now.)

$$\frac{dT_c}{dt} = \frac{T_h - T_c}{\tau_s} - \frac{\dot{Q}_s}{C_s}$$

τ_s and C_s apply to the steam generator including the mass of water within the cold side piping. The values of $P_r\ and\ \dot{Q}_s$ are positive.

We will ignore the change in water density that this change in temperature creates. So, we will directly relate system volume with water mass.

So, with these concepts put together, we can write the following matrix equations for our coupled reactor, steam generator system.

$T = \begin{bmatrix} T_h \\ T_c \end{bmatrix}$	$A = \begin{bmatrix} -1/\tau_r & 1/\tau_r \\ 1/\tau_s & -1/\tau_s \end{bmatrix}$	$B = \begin{bmatrix} \frac{P_r}{C_r} \\ -\frac{\dot{Q}_s}{C_s} \end{bmatrix}$
$\frac{dT}{dt} = AT + B$	$T(t) = e^{At}T_0 + e^{At}\int_0^t e^{-At'} * B\,dt'$	

A is singular, the eigenvalues are zero and $-1/\tau_{eff}$ where $\tau_{eff} = \frac{\tau_r\tau_s}{\tau_r+\tau_s}$. The steady state difference in the two temperatures will be proportional to the sum of the reactor and steam generators (both taken as positive).

$$\Delta T = \tau_{eff}(\frac{P_r}{C_r} + \frac{\dot{Q}_s}{C_s})$$

The steady state average temperature with $P_r = \dot{Q}_s$ is best found using conservation of energy:

$$T_{Ave} = T_{Ave}^{initial} + \frac{\tau_{eff}}{2}\left(\frac{P_r}{C_r} - \frac{\dot{Q}_s}{C_s}\right) = T_{Ave}^{initial} + \frac{\tau_{eff}P_r}{2}\left(\frac{1}{C_r} - \frac{1}{C_s}\right)$$

The heat capacities and the time constants are proportional to mass. We may define the fraction of the total mass of fluid that is within the reactor as μ, and we have:

$$\tau_r = \mu\tau_{0,}\ \tau_s = (1-\mu)\tau_{0,}\ C_r = \mu C_{0,}\ C_s = (1-\mu)\ C_{0,}$$

Using these definitions, the average temperature becomes as derived below:

$$T_{Ave} = T_{Ave}^{initial} - \frac{\tau_0 P_r(2\mu - 1)}{2C_0}$$

The steady, state mass weighted average temperature will be constant. This is what causes the Tave to droop or rise when the hot and cold legs have different volumes:

$$\mu T_h + (1-\mu)T_c = T_{Ave}^{initial}$$

$$\mu T_h + (1-\mu)T_c - \frac{T_h + T_c}{2} = T_{Ave}^{initial} - T_{Ave}^{final}$$

$$\mu(T_h - T_c) + T_c - \frac{T_h + T_c}{2} = T_{Ave}^{initial} - T_{Ave}^{final}$$

$$\Delta T = \tau_{eff}\left(\frac{P_r}{C_r} + \frac{\dot{Q}_s}{C_s}\right) = \frac{\mu(1-\mu)\tau_0 P_r}{\mu + (1-\mu)} * \left(\frac{1}{\mu C_0} + \frac{1}{(1-\mu)C_0}\right) = \frac{\tau_0 P_r}{C_0}$$

$$\mu\Delta T - \frac{\Delta T}{2} = T_{Ave}^{initial} - T_{Ave}^{final}$$

$$\frac{(2\mu - 1)}{2}\tau_0 P_r / C_0 = T_{Ave}^{initial} - T_{Ave}^{final}$$

This is equivalent to saying that if the heat flow in and out of the system is balanced, the total energy in the system remains constant.

For the case where the reactor and steam generator are both constant loads (balanced or not), the two temperatures may be computed as follows:

$\tau_{eff} = \frac{\tau_r \tau_s}{\tau_r + \tau_s}$	$U = \frac{I}{\tau_{eff}} + A$	$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$e^{At} = I + \tau_{eff}(1 - e^{-t/\tau_{eff}})A$
$B = \begin{bmatrix} \frac{P_r}{C_r} \\ -\frac{\dot{Q}_s}{C_s} \end{bmatrix}$	$K(t) = \int_0^t e^{-At'} * B\, dt' = \left[t\,\tau_{eff}U + {\tau_{eff}}^2\left(e^{\frac{t}{\tau_{eff}}} - 1\right)I\right]AB$		
$\begin{bmatrix} T_h \\ T_c \end{bmatrix} = e^{At}\begin{bmatrix} T_h \\ T_c \end{bmatrix}_0 + e^{At}K(t)$			

This equation will have serious numerical problems computationally due to the positive exponent in the K(t) factor. This may be remedied using the following equivalent formulation.

$(I + \tau_{eff}A)A = 0$	$\begin{bmatrix} T_h \\ T_c \end{bmatrix} = e^{At}\begin{bmatrix} T_h \\ T_c \end{bmatrix}_0 + t\,\tau_{eff}UB) - {\tau_{eff}}^2 * \left(1 - e^{-\left(\frac{t}{\tau_{eff}}\right)}\right)AB$

If the heat flows between the steam generator and reactor are balanced, then the factor $UB = 0$ so the term that is linear in t will vanish. In this balanced case the final $T_{Ave}^{final} = T_{Ave}^{initial} - [0.5,0.5] * {\tau_{eff}}^2 AB.$ This simplifies to the same equation for the steady state average temperature that we found above.

The expression for e^{At} above was developed by finding the eigenvalues, $\{d_i\}$ and modal matrix, M, related to the A matrix and then following reverse diagonalization, (See Note 6):

$$e^{At} = M\ diagonal(e^{d\,t})\ M^{-1}$$

The equation K(t) represents the integral factor:

$$\int_0^t e^{-At\prime} * B\ dt'$$

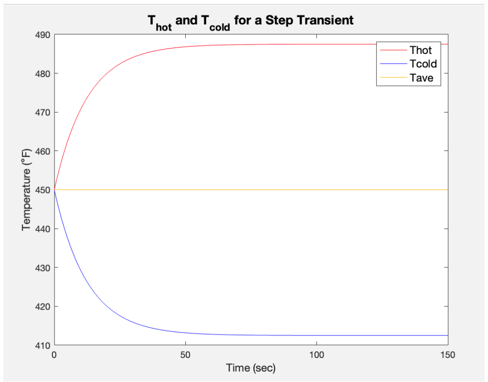
The integral term has been computed assuming that both heat flow rates are constant in time. If this is not the case, K(t) will need to be recomputed. Performance of this integral is greatly simplified by using the form developed for e^{At} above.

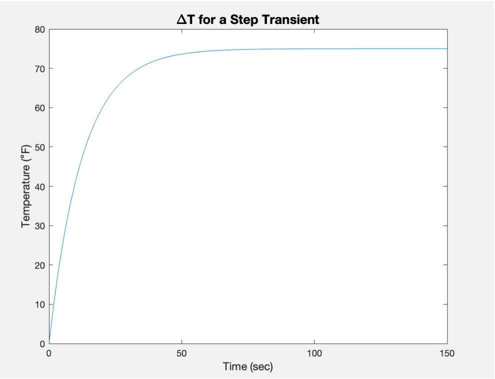
Example:

Consider a system where $\Delta T^{100\%} = 75\ ^\circ\mathrm{F},\ \tau_0 = 50$. With full power being 100% the base heat capacity, $C_{0,} = 100\% * \frac{\tau_o}{\Delta T^{100\%}} = 66.66\ [\% \ - \mathrm{sec}/^\circ\mathrm{F}]$.

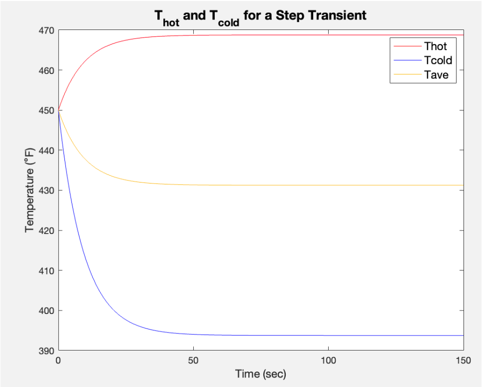
Using the equation above we find the following.

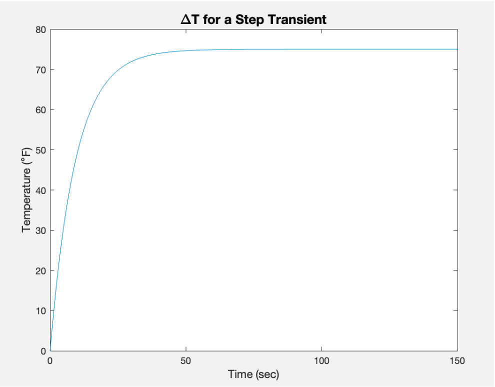
Equal Masses





Unequal Masses
$\mu = 0.75$





Loop Transport Time

This presentation has ignored the effects of loop transport time within the reactor/steam generator system. This may be modeled effectively as a time delay in both legs of the system. This phenomenon is discussed in “Note 3” at the end of this document. The impact of the delays is discussed and then analyzed using Laplace (frequency domain) analysis.

MATLAB Code

What follows are MATLAB codes that can make plots for the hot, cold, and average temperatures in this problem. They also plot the difference between Th and Tc.

The first version of this code uses the equations that we have developed for the various steps in this effort. The second version uses a MATLAB ODE solver to directly integrate the differential equations. The two methods produce the same result.

```
% LoopTimeDomainPowerStepTransient.m
% William Locke
% July 2025
%
% This code demonstrates the temperature performance of a
% coupled heat transfer system with unequal water mass loops.
% The 2 by N matrix holds the hot and cold temperatures
% as the transient happens.
function LoopTimeDomainPowerStepTransient()
    N = 1000;
    Tfinal = 150;
    Pr = 100;  Q_dot = 100;
    RMF = 0.75;  % Fraction of the water that is in the reactor.
    Tau = 50; Tau_r = RMF*Tau; Tau_s = (1-RMF)*Tau;
    C = 2/3*100;  Cr = RMF*C;  Cs = (1-RMF)*C;
    Teff = Tau_r*Tau_s/(Tau_r+Tau_s);
    B = [Pr/Cr;-Q_dot/Cs];
    A = [-1/Tau_r, 1/Tau_r;1/Tau_s, -1/Tau_s];
    U = eye(2)/Teff+A;
    T = zeros(2,N);
    To(:,1) = [450;450];
    t = zeros(N,1);
    for i=1:N
        t(i) = (i-1)/(N-1)*Tfinal;
        EAT = eye(2)+Teff*(1-exp(-t(i)/Teff))*A;
        T(:,i)= EAT*(To+ t(i)*Teff*U*B) -...
            Teff^2*(1-exp(-t(i)/Teff))*A*B;
    end
    figure
    plot(t,T(1,:),'r',t,T(2,:),'b',t,(T(1,:)+T(2,:))/2,'--k')
    title("T_{hot}, T_{cold}, and T_{Ave} for a Step Transient");
    xlabel("Time (sec)");
    ylabel("Temperature \circF");
    legend("Thot","Tcold","Tave")
    figure
    plot(t,T(1,:)-T(2,:))
    title("\DeltaT for a Step Transient");
    xlabel("Time (sec)");
    ylabel("Temperature \circF");
    Final_Delta_T_Ave = 450+(T(1,end)+T(2,end))/2 -(To(1)+To(2))/2

end
```

```matlab
% LoopTimeDomainPowerStepTransientODE.m
% William Locke
% July 2025
%
% This code demonstrates the temperature performance of a
% coupled heat transfer system with unequal water mass loops.
% The 2 by N matrix holds the hot and cold temperatures
% as the transient happens.
function LoopTimeDomainPowerStepTransientODE()
    N = 1000;
    Tfinal = 150;
    Pr = 100;  Q_dot = 100;
    RMF = 0.75;  % Fraction of the water that is in the reactor.
    Tau = 50; Tau_r = RMF*Tau; Tau_s = (1-RMF)*Tau;
    C = 2/3*100;  Cr = RMF*C;  Cs = (1-RMF)*C;
    Teff = Tau_r*Tau_s/(Tau_r+Tau_s);
    B = [Pr/Cr;-Q_dot/Cs];
    A = [-1/Tau_r, 1/Tau_r;1/Tau_s, -1/Tau_s];
    function dTdt = TargetFunction(t,T)
        dTdt = A*T+B;
    end
    T = zeros(2,N);
    To(:,1) = [450;450];
    [t,Thist] = ode45(@TargetFunction,[0,Tfinal],To(:,1));
    T = Thist';
    figure
    plot(t,T(1,:),'r',t,T(2,:),'b',t,(T(1,:)+T(2,:))/2,'--k')
    title("T_{hot}, T_{cold}, and T_{Ave} for a Step Transient");
    xlabel("Time (sec)");
    ylabel("Temperature \circF");
    legend("Thot","Tcold","Tave")
    figure
    plot(t,T(1,:)-T(2,:))
    title("\DeltaT for a Step Transient");
    xlabel("Time (sec)");
    ylabel("Temperature \circF");

    Final_Delta_T_Ave = (T(1,end)+T(2,end))/2 -(To(1)+To(2))/2
end
```

3. Steam and Reactor Power Dynamics in a Thermal Reactor

Time Spent on the Big Picture

What happens when we draw more steam off a reactor (increase steam flow)

Start in steady state Reactor Power = Steam demand, Temperature is constant. Pressurizer level is Constant

- Open the Main Steam Valve
- Flow in the steam system goes up. Steam Pressure Drops
- More boiling happens in the steam generator because the pressure dropped.
- Average Temperature in the Steam Generator (steam side) goes down.
- Heat transfer across the Steam Generator tubes goes up.
- Tc drops, Th is the same for the moment.
- When the colder water reaches the reactor, positive reactivity is inserted.
- Reactor Power goes up which increases Th, this raises Tave.
- Steady state will have a larger delta T, (Th-Tc), reactor power will match steam demand.
- If we are using high enrichment fuel, the Tave will be recovered to its original value. If we are using low enrichment fuel, Tave will drop due to doppler broadening. A boron or rod adjustment will be needed.

The pressurizer will have had an out surge during this transient.
The power may or may not have damped oscillation around the final value.

Whenever reactor power is greater than steam demand Tave will go up.
Whenever reactor power is less than steam demand Tave will go down.

Nonlinear Equations

The effect of a time variable reactivity is that the solution will not have the form e^{At} as before. That solution required linear equations without time variation in the coefficients.

We can still think in terms of a matrix and vector based coupled system. The solvers that we will use will be mathematical methods that will be chosen based of the numerical difficulties of the problem. If the prompt jump assumption is used and the reactivity is far from β we will generally get good results. This makes an eighth order system of coupled equations (6 – reactor kinetics, and two temperature).

When using MATLAB, the function ode45()(with 'RelTol' set to 1e-6) is usually sufficient for making this solution method with the prompt jump assumption. If Numerical noise appears in the data ode15s() is recommended.

As an alternative to this approach, one could also approximate the solutions to these equations by using a home grown fourth order Runge-Kuta or even by using an iterate solution of the following form.

$$X_k = (I - (A/2)\tau_{step}^{[sec]})^{-1}((I + (A/2)\tau_{step}^{[sec]})X_{k-1} + \Gamma\tau_{step}^{[sec]})$$

But in any, case A would need to be recomputed for each k or nearly each k depending on the step size and the required precision. This is costly and not as dependable as using the MATLAB built in ODE solvers. The major advantage of the MATLAB solvers is that they vary the step size to optimize execution given the rate of change of the solution.

Modeling plant behavior with Reactor Kinetics and Steam Flow

We will consider the effect of sources (S) to be unimportant for this section.

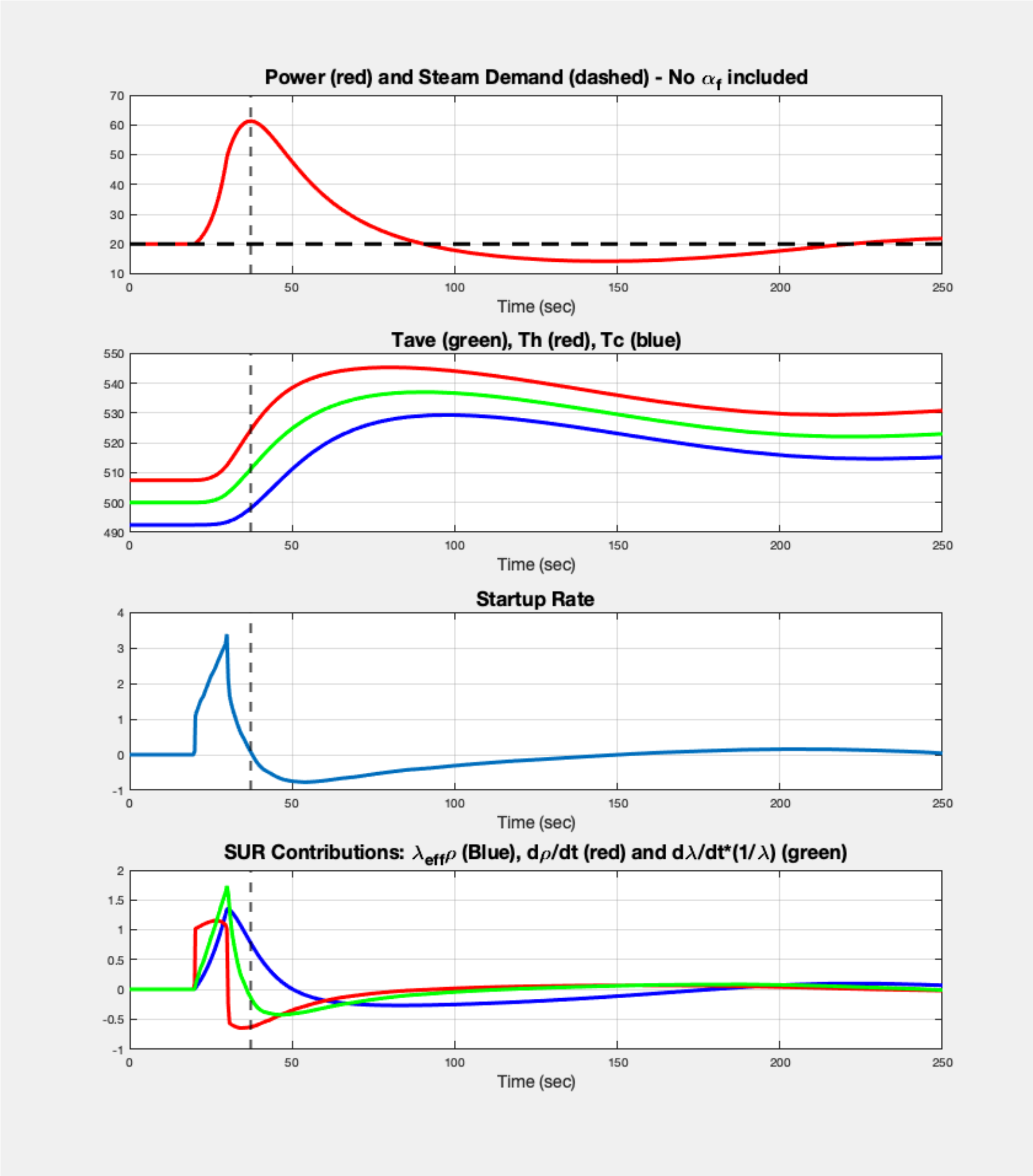
One line example: Prompt Jump Power -> $\mathrm{P} = \frac{\Lambda \sum_i \lambda_i C_i(t)}{(\beta - \rho(t))}$

$$\frac{dC_1}{dt} = \frac{\beta_1 P}{\Lambda} - \lambda_1 C_1 = \frac{\beta_1 \Lambda \sum_i \lambda_i C_i(t)}{\Lambda\big(\beta - \rho(t)\big)} - \lambda_1 C_1 = \{\sum_i \beta_1 \lambda_i C_i(t) - \lambda_1 C_1\big(\beta - \rho(t)\big)\big)\}/\big(\beta - \rho(t)\big)$$

<table>
<tr>
<td>$X(t) = \begin{vmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ T_h \\ T_c \end{vmatrix}$</td>
<td colspan="2">$Xo(0) = \begin{vmatrix} \beta_1/\lambda_1 \\ \beta_2/\lambda_2 \\ \beta_3/\lambda_3 \\ \beta_4/\lambda_4 \\ \beta_5/\lambda_5 \\ \beta_6/\lambda_6 \\ 0 \\ 0 \end{vmatrix} \frac{N_0}{\Lambda}$</td>
<td>$\Gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \frac{SGPower}{C_s}$</td>
</tr>
<tr>
<td colspan="4">$$A = \left(\frac{1}{\beta - \rho}\right) \begin{vmatrix} \beta_1\lambda_1 - \lambda_1(\beta-\rho) & \beta_1\lambda_2 & \beta_1\lambda_3 & \beta_1\lambda_4 & \beta_1\lambda_5 & \beta_1\lambda_6 & 0 & 0 \\ \beta_2\lambda_1 & \beta_2\lambda_2 - \lambda_2(\beta-\rho) & \beta_2\lambda_3 & \beta_2\lambda_4 & \beta_2\lambda_5 & \beta_2\lambda_6 & 0 & 0 \\ \beta_3\lambda_1 & \beta_3\lambda_2 & \beta_3\lambda_{3-\lambda_3(\beta-\rho)} & \beta_3\lambda_4 & \beta_3\lambda_5 & \beta_3\lambda_6 & 0 & 0 \\ \beta_4\lambda_1 & \beta_4\lambda_2 & \beta_4\lambda_3 & \beta_4\lambda_4 - \lambda_4(\beta-\rho) & \beta_4\lambda_5 & \beta_4\lambda_6 & 0 & 0 \\ \beta_5\lambda_1 & \beta_5\lambda_2 & \beta_5\lambda_3 & \beta_5\lambda_4 & \beta_5\lambda_5 - \lambda_5(\beta-\rho) & \beta_5\lambda_6 & 0 & 0 \\ \beta_6\lambda_1 & \beta_6\lambda_2 & \beta_6\lambda_3 & \beta_6\lambda_4 & \beta_6\lambda_5 & \beta_6\lambda_6 - \lambda_6(\beta-\rho) & 0 & 0 \\ \Lambda\lambda_1/C_r & \Lambda\lambda_2/C_r & \Lambda\lambda_3/C_r & \Lambda\lambda_4/C_r & \Lambda\lambda_5/C_r & \Lambda\lambda_6/C_r & \frac{-(\beta-\rho)}{\tau_r} & \frac{(\beta-\rho)}{\tau_r} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(\beta-\rho)}{\tau_s} & \frac{-(\beta-\rho)}{\tau_s} \end{vmatrix}$$</td>
</tr>
<tr>
<td rowspan="5">$$\frac{dX}{dt} = AX + \Gamma$$</td>
<td colspan="3">Compute ρ and P with the following equations:</td>
</tr>
<tr>
<td>$P(t) = \frac{G(t)}{\beta - \rho(t)}$</td>
<td>$\rho(t) = \rho^{other}(t) + \alpha_f(P(t) - P(0))$</td>
<td>$c = \frac{\Lambda \sum_i \lambda_i C_i(t)}{\alpha_f}$</td>
</tr>
<tr>
<td rowspan="2">$\mathrm{d}T_{ave} = (T_{ave} - T_0)$</td>
<td>$b = (-\beta + \alpha_w \mathrm{d}T_{ave} + \rho^{other} - \alpha_f P(0))/\alpha_f$</td>
<td rowspan="2"></td>
</tr>
<tr>
<td>$P(t) = (-b + \sqrt{b^2 - 4c})/2$</td>
</tr>
<tr>
<td colspan="3">P_0 is a reference power, and ρ^{other} is the reactivity due to effects other than power. This includes water temperature reactivity $\rho^{water} = \alpha_w(T_{ave} - T_0)$, $T_{ave} = (T_h + T_c)/2$</td>
</tr>
</table>

Example Transients Computed with MATLAB based on these equations:

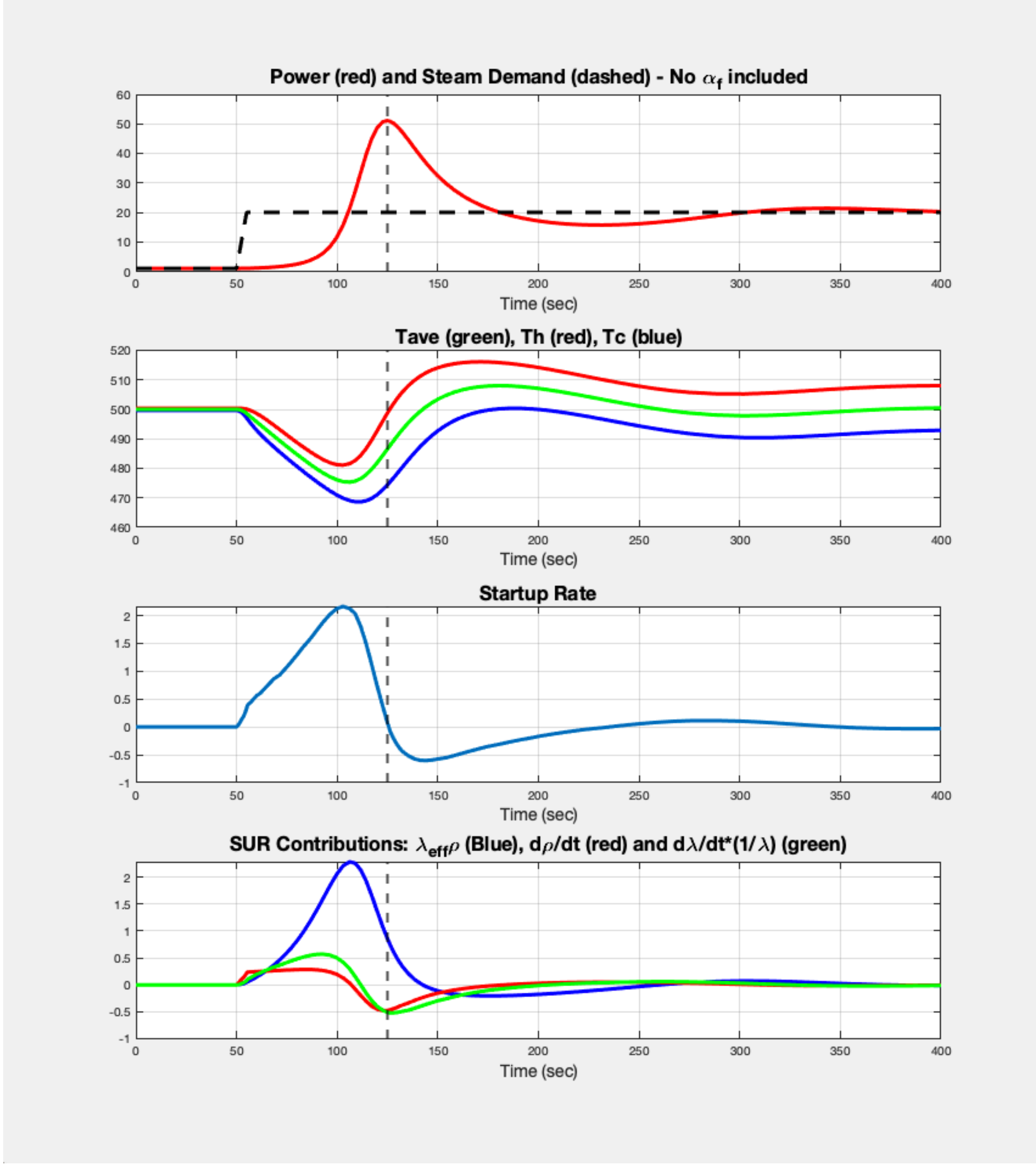
- The Fuel Reactivity power coefficient is – 10 pcm/percent.
- The effective Beta is taken as simply the natural Beta.
- The initial T_{ave} is taken as 500 degrees F.
- The target full power steady state ΔT, (T_h-T_c), is set as 75 degrees F. The temperatures are referenced to T_{ave} so at zero power $T_h = T_c = 0$;
- Plots with and without fuel temperature feedback are provided.
- The water temperature coefficient is -10 pcm/degree. Pipe transport delay time is not considered. The hot and cold plenum $\boldsymbol{\tau's}$ are 25 sec.
- Ambient heat loss and pump heating are ignored



<u>Rod Pull at 20% Power yielding 250pcm over 10 seconds</u>

The power rose and then came back to its original value. Temperature rose while power exceeded steam demand. The final temperature rise is 25°F. The 20% power ΔT is 15°F. Notice the jump in Sur on the rod pull. The pull ramps from time = 20 sec to time = 30 sec. The initial jump in SUR is $26.06dpm - \sec\left(\frac{250/10sec}{640}\right) = 1.02dpm.$

Steam Demand is Raised from 1% to 20%



Power (red) and Steam Demand (dashed) - No α_f included

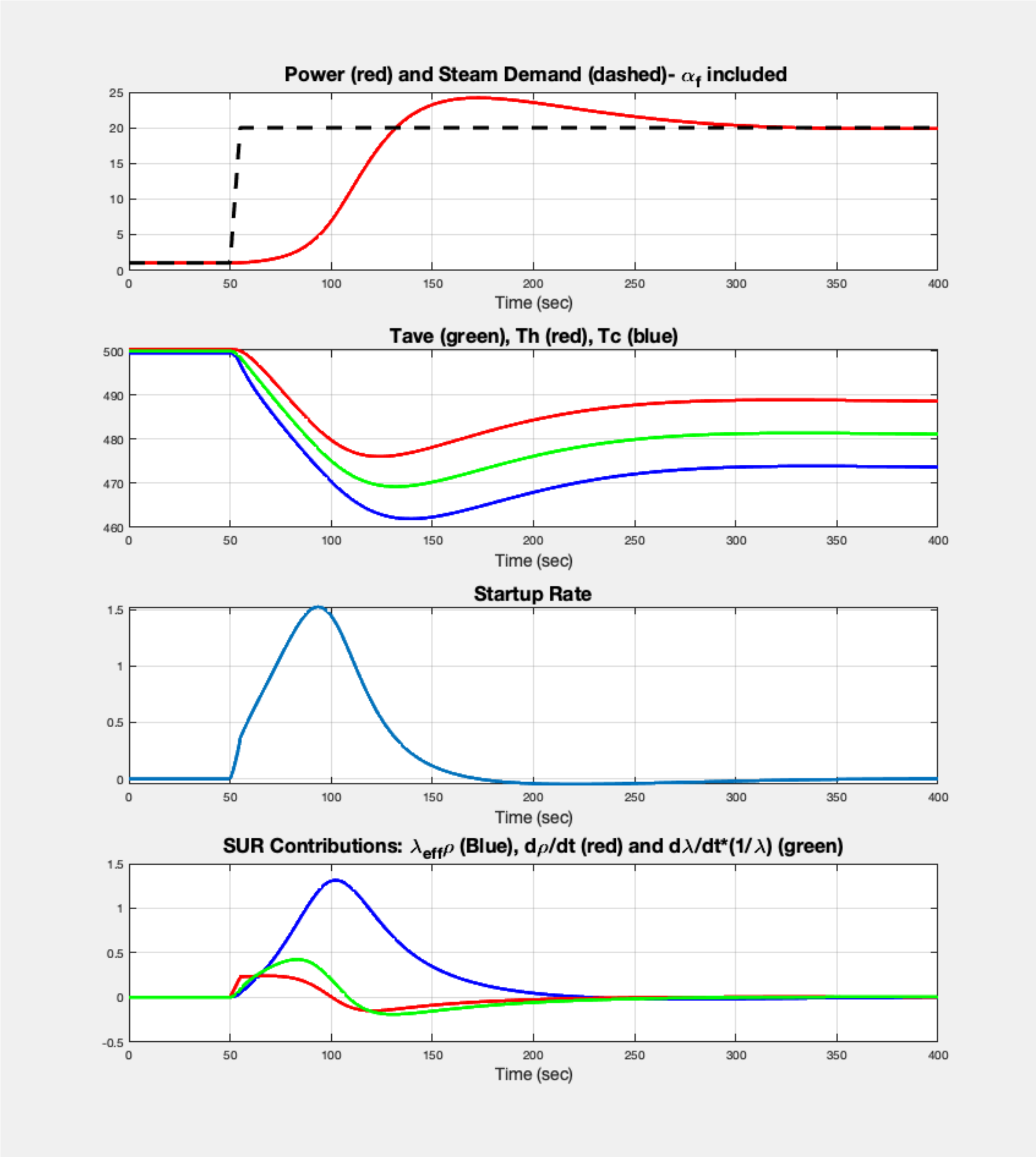
Tave (green), Th (red), Tc (blue)

Startup Rate

SUR Contributions: $\lambda_{eff}\rho$ (Blue), $d\rho/dt$ (red) and $d\lambda/dt^*(1/\lambda)$ (green)

<u>Steam Demand Change 1% to 20% No Fuel Temperature Feedback</u>

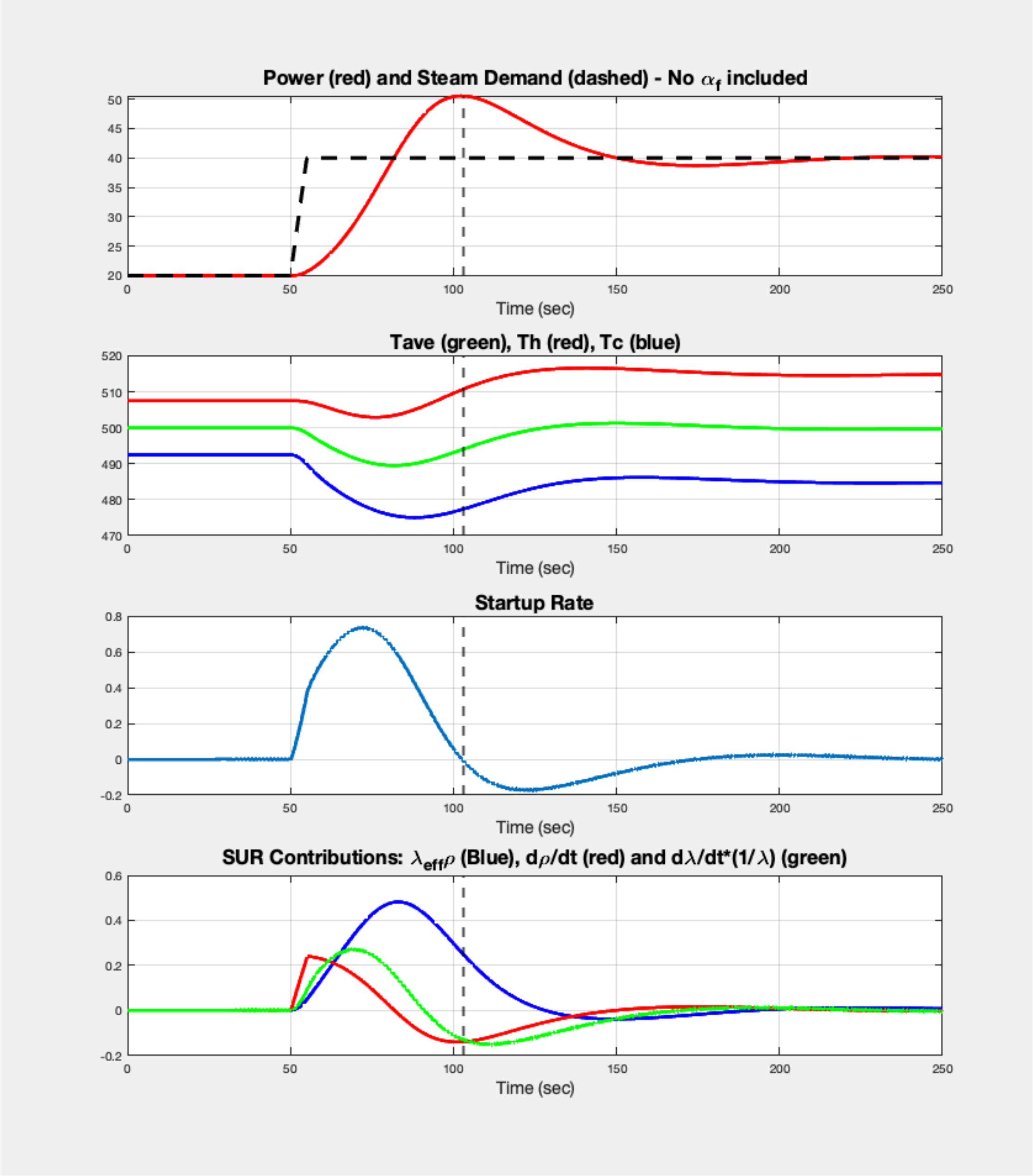
This is a ramp increase in steam demand, starting at a low power. The response includes an initial drop in T_c, T_{Ave}, and T_h - followed by a quick overshoot in power. Following that power settles out at the new steam demand. Look at the SUR contributions at the point of power turning. Note that it is zero at that point. Look at T_{Ave} at the point of power turning. What does this tell you about the reactivity?



Steam Demand Change 1% to 20% with Fuel Temperature Feedback

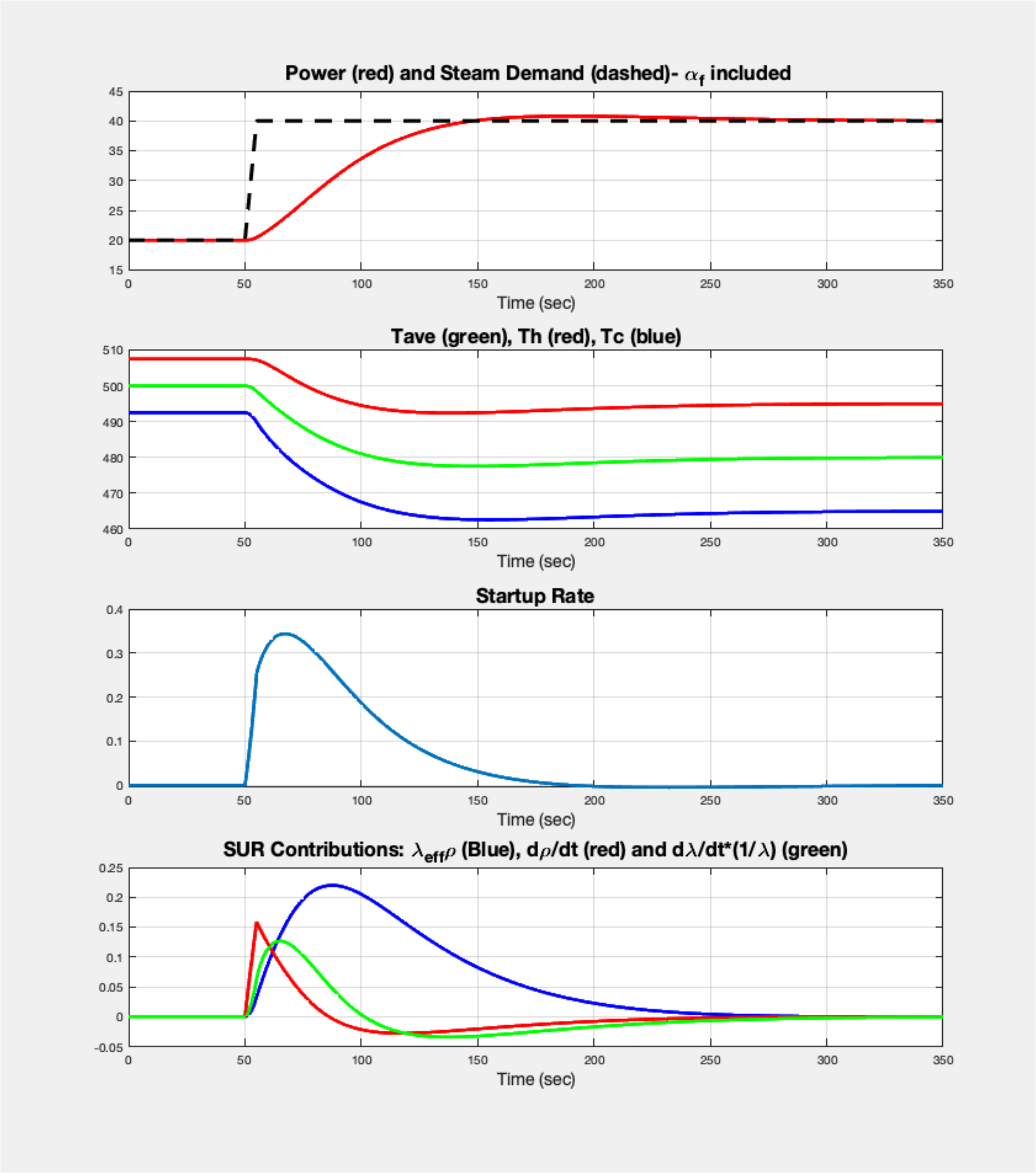
This is the same transient as the last one but with fuel power reactivity feedback included in the model. The feedback slows down the response. As you would expect, the T_{Ave} droops in response to the power increase by 20.0°F.

Steam Demand is Raised from 20% to 40%



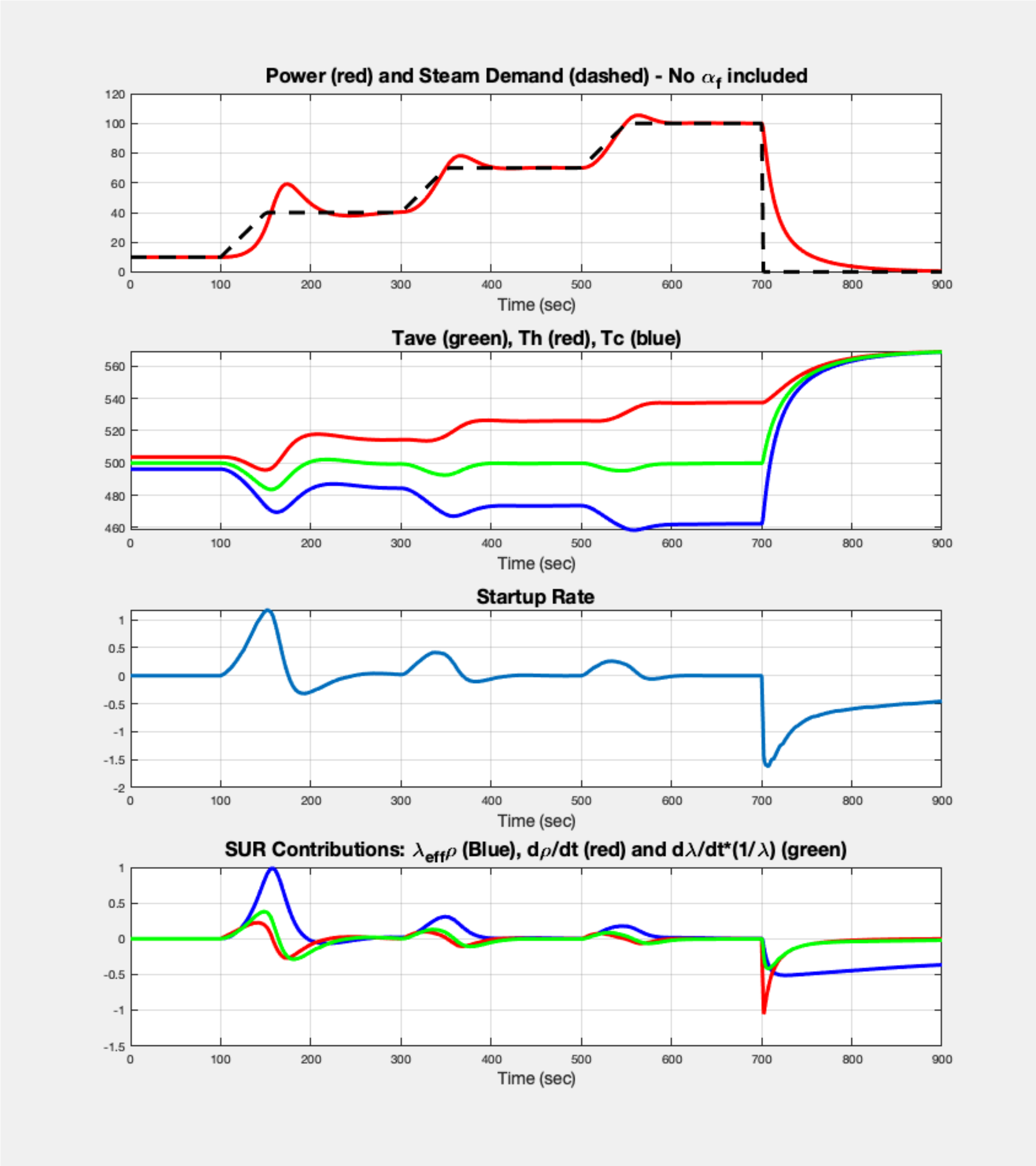
<u>Steam Demand Change 20% to 40% with No Fuel Temperature Feedback</u>

This is another 20% steam demand change - the results are quite different. The power more closely tracks with steam demand and the overshoot is reduced. The difference is that we ended this 20% power change at a higher power. In the next section of this paper we shall see that at higher powers there is more damping.



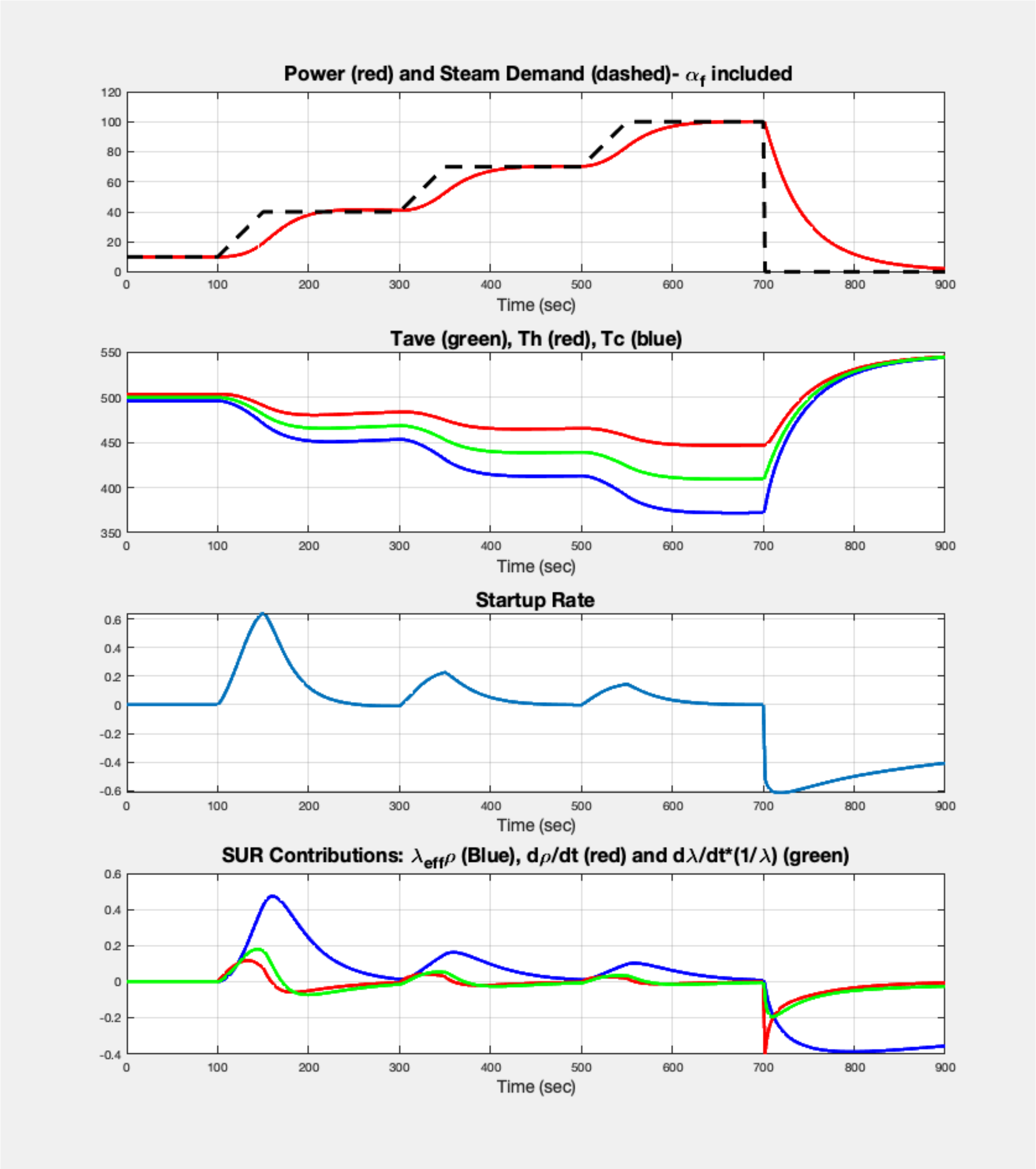
Steam Demand Change 20% to 40% with Fuel Temperature Feedback

Notice how the feedback has caused a slowing in the response. The shift in T_{Ave} is the same as in the previous case.



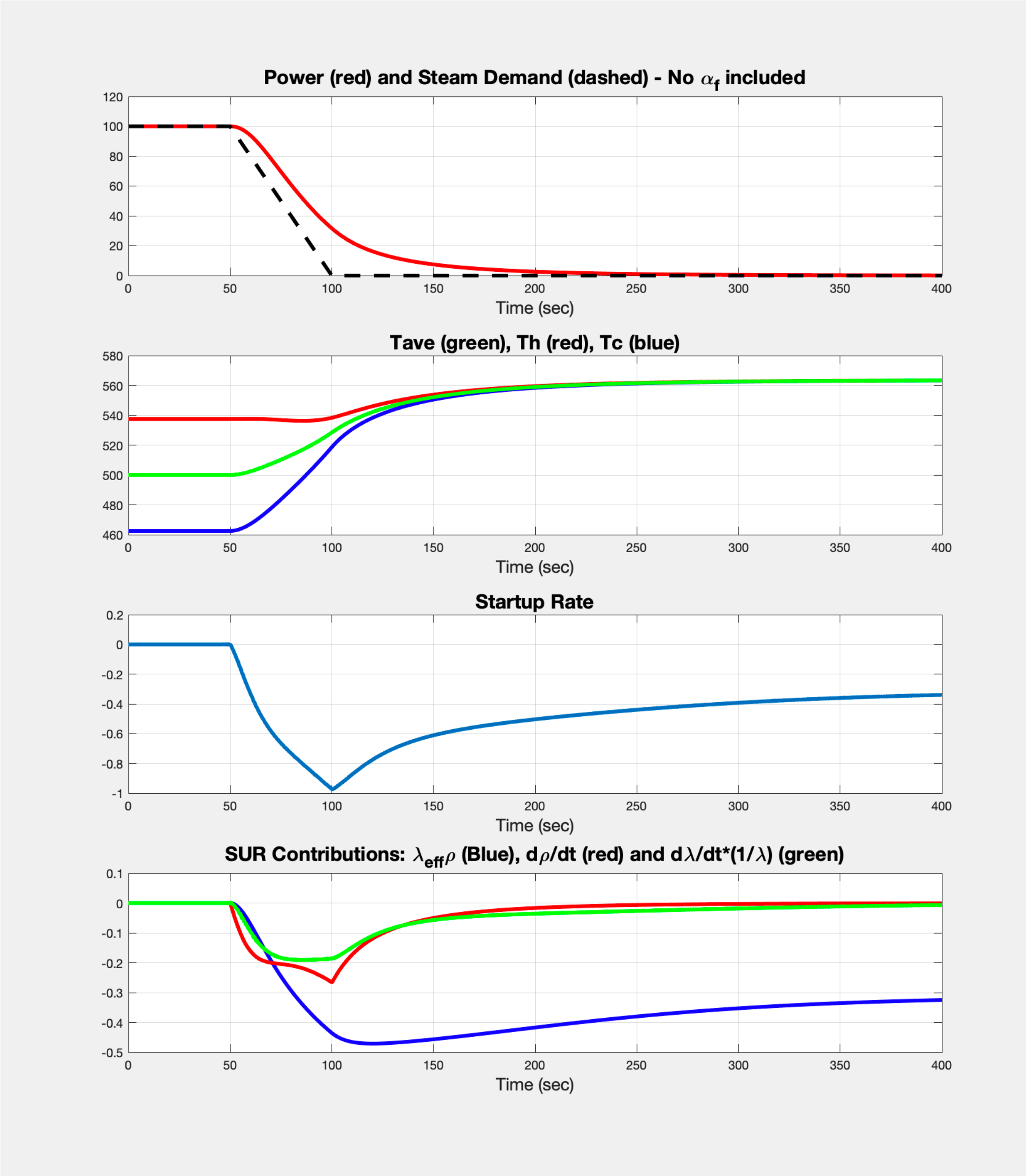
Multiple Ramps in Steam Demand Followed by Shutdown – No Fuel Feedback.

Steam Demand ramps from 10% to 40%, 40% to 70%, and 70% to 100%, followed by a Turbine trip. Why did the temperature go up at the end of the transient? Why did the temperatures converge? Why did we get the sudden drop in SUR at 700 sec?



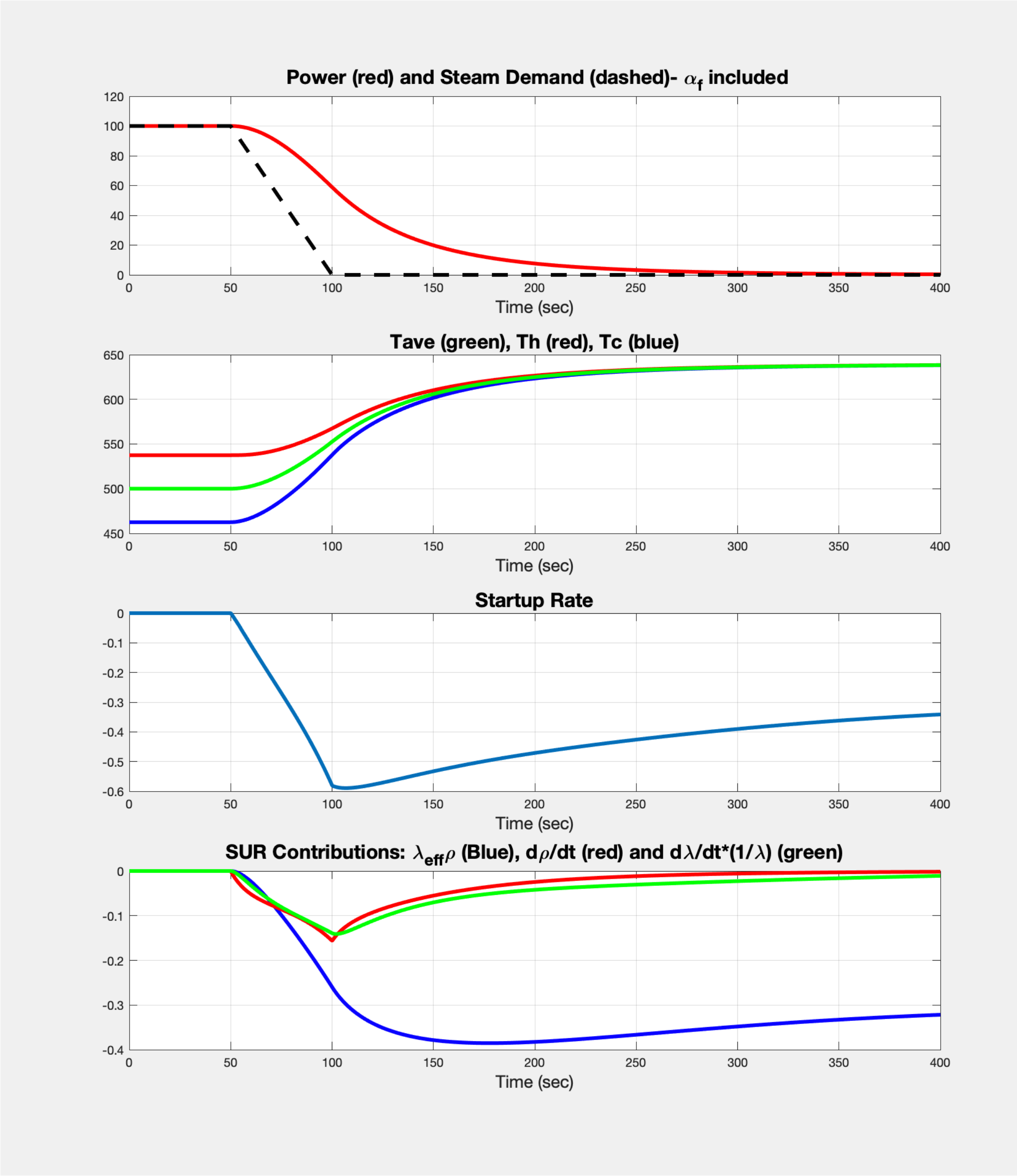
Multiple Ramps in Steam Demand Followed by Shutdown – With Fuel Feedback.

Steam Demand ramps from 10% to 40%, 40% to 70%, and 70% to 100%, followed by a Turbine trip. This case includes power reactivity feedback. Comment on the final temperature – is this realistic? What problems do you see?



Shutdown From 100% - Without Fuel Feedback.

Notice the apparent steady state SUR after about ten minutes. Why does the initial SUR transient look the way it does? Why is T_{Ave} in the final case higher than the initial temperature? Comment on the final temperature – is this realistic?



Shutdown From 100% - With Fuel Feedback.

Contrast this with the previous case. Why does the final T_{Ave} wind up as it does? Comment on the final temperature – is this realistic? What problems do you see?

Power Turning Revisited

Consider a problem the following statement and results:

A plant has a steady startup rate of 1 dpm below the point of adding heat. How much reactivity is present? How much reactivity will remain when power turns if the heat-up-rate is 0.092 °F/sec at its peak? For this case $\alpha_T = -20\ pcm/°\mathrm{F}$

Assume: $\beta = 640\ pcm\ and\ \lambda_{eff} = \frac{0.1}{sec}$. Ignore sources and changes in λ_{eff}.

$SUR = 26.06\ dpm - sec\left\{\frac{\dot{\rho} + \lambda_{eff}\rho}{\beta - \rho}\right\}$	$\rho = \frac{\frac{SUR*\beta}{(26.06dpm - sec)} - \dot{\rho}}{\lambda_{eff} + \frac{SUR}{(26.06dpm - sec)}}$
$\dot{\rho} = 0, \qquad SUR = 1\ dpm,$ $\lambda_{eff} = 0.1/sec$	$\rho = -\dot{\rho}/\lambda_{eff}$
$\dot{\rho} = \alpha_T * 0.092° \frac{\mathrm{F}}{\mathrm{sec}} = -1.84\ \frac{pcm}{sec}$ SUR = 0.0 dpm	$\rho = 18.46\ pcm$ At the point of power turning. Initially the reactivity was 177.48 pcm to get the one DPM SUR.

We will now reconsider this problem using the kinetics solution applying the prompt jump assumption as above. A rod pull is used to establish a one decade per minute startup. This requires a reactivity of approximately $0.25\beta = 160\ pcm$ rather than approximately 0.28β required using the $\lambda_{eff} = \frac{0.1}{sec}$. The actual λ_{eff} (prior to the heat up beginning) rises to about $0.118\ \frac{1}{sec}$.

Once the power is high enough to cause heating the changing water temperature reduces the net positive reactivity, and it introduces a $\dot{\rho}$ which also impacts the startup rate. In this case the power turns with a reactivity of 46.9 pcm. This is significantly higher than the value found above. This is because the standard startup equation ignores the time rate of change of λ_{eff}.

We derived the following equation:

$$SUR = 26.06\,[dpm - sec]\left[\frac{\dot{\rho} + \lambda_{eff}\rho}{(\beta - \rho)} + \frac{\dot{\lambda_{eff}}}{\lambda_{eff}}\right]$$

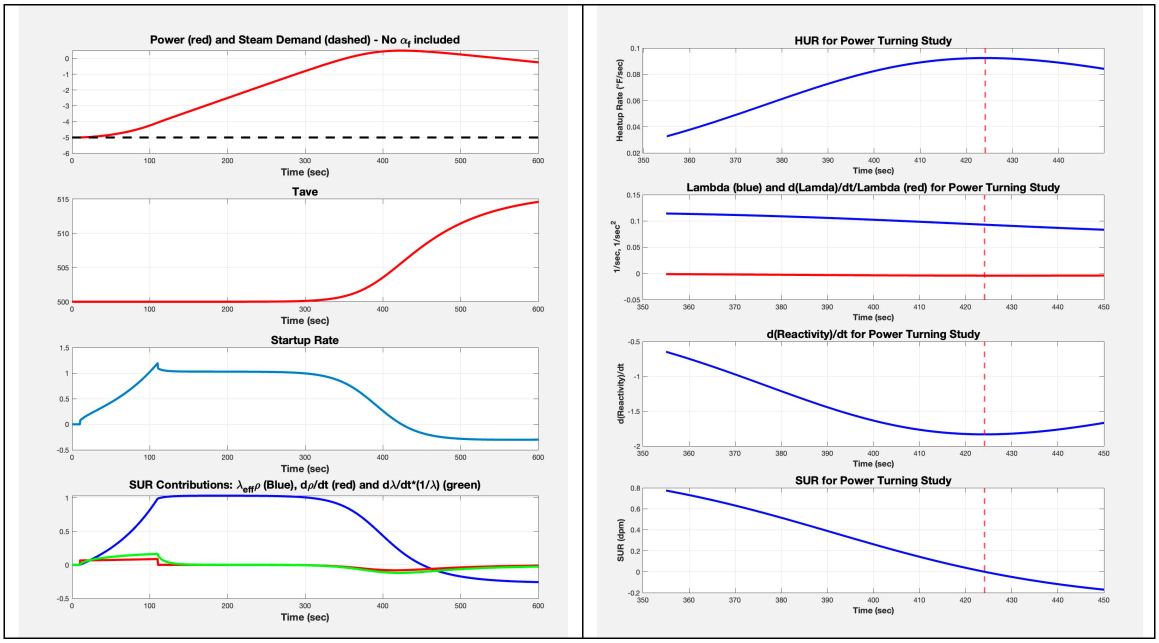
The kinetics solution may be used to find this ratio at the point of power turning. The value $\frac{\dot{\lambda_{eff}}}{\lambda_{eff}} = -0.004256\ 1/\text{sec}$. Also. at the peak power $\lambda_{eff} = 0.0929\ 1/sec$.

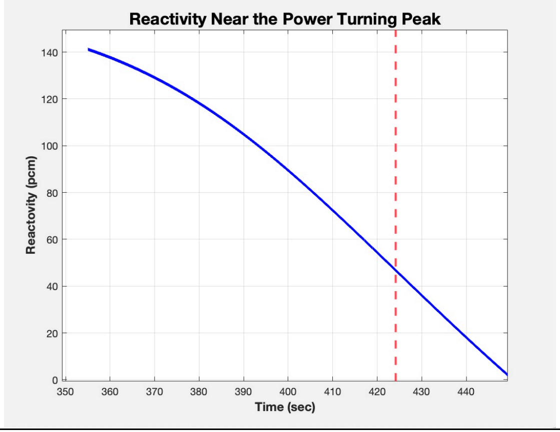
We may use the equation for reactivity again as above with the startup rated modified as:

$SUR' = SUR - 26.06\,(dpm - \text{sec}) * \frac{\dot{\lambda_{eff}}}{\lambda_{eff}}$	$\rho = \frac{\frac{SUR' * \beta}{(26.06dpm - sec)} - \dot{\rho}}{\lambda_{eff} + \frac{SUR'}{(26.06dpm - sec)}}$

So, at the peak, SUR would become 0.1109 dpm. Using the equation for the reactivity at the peak we obtain 47.04 pcm which is much closer to the kinetics equation result.

The following plots support the power turning discussion.



Reactivity Near the Power Turning Peak
Reactovity (pcm)
0
20
40
60
80
100
120
140
350
360
370
380
390
400
410
420
430
440
Time (sec)

Stability Analysis

The stability of a system may be examined near a fixed point using a perturbation theory approach. It is important to understand that unlike a linear system, a nonlinear system may not be globally stable because it is stable at one fixed point. For our case we will consider an initial steady state condition as a fixed point examining the behavior of the system about that point when we disturb it. This could be something like moving the power off its steady state value. Our systems will be stable, but the resulting eigenvalues will vary with initial power. We can also see how the perturbation evolves in time.

Start this analysis using a six delayed group model with a single $T_{ave.}$
We are ignoring fuel feedback for this analysis.

$$P = \frac{\Lambda \sum_{i=1}^{6} \lambda_i C_i}{\beta - \rho}$$

$$\frac{dT_{ave}}{dt} = (P_{Rx} - \dot{Q}_{SG})/c$$

$$\frac{dC_j}{dt} = \frac{\beta_j}{\Lambda} P - \lambda_j C_j$$

Now suppose that our steady state has $\rho_0 = 0, \frac{dT_{ave\_0}}{dt} = 0,\ \frac{dC_{j\_0}}{dt} = 0$ and let each variable parameter take the form $Para = Para_0 + \delta Para$:

$$(\mathrm{P}_0 + \delta P)(\beta - \alpha_w \delta T_{ave}) = \Lambda \sum_{i=1}^{6} \lambda_i (C_{i\_0} + \delta C_i)$$

And from the steady state condition this becomes:

$$\delta P \beta - P_0 \alpha_w \delta T_{ave} = \Lambda \sum_{i=1}^{6} \lambda_i \delta C_i$$

Because:

$$\mathrm{P}_0 \beta = \Lambda \sum_{i=1}^{6} \lambda_i C_{i\_0}$$

We have also ignored a product of two perturbations: $\delta P\ \delta T_{ave} \approx 0$

Solving this for the perturbation in power we get:

$$\delta P = 1/\beta [\Lambda \textstyle\sum_{i=1}^{6} \lambda_i \delta C_i + P_0 \alpha_w \delta T_{ave}]$$

Now substitute this into the equations for $\dot{C_\iota}, \dot{T}$:

$$\dot{C_j} = \frac{\beta_j[\sum_{i=1}^{6} \lambda_i \delta C_i + P_0 \alpha_w \delta T_{ave}]}{\beta} - \lambda_j \delta C_j$$

$$\dot{T} = \frac{\frac{[\Lambda \sum_{i=1}^{6} \lambda_i \delta C_i + P_0 \alpha_w \delta T_{ave}]}{\beta} - \dot{Q}_{SG}}{C}$$

While these equations look dreadful, we can put them into a simple matrix form as follows:

$$\Psi = \frac{1}{\beta}\begin{bmatrix} (\beta_1 - \beta)\lambda_1 & \beta_1\lambda_2 & \beta_1\lambda_3 & \beta_1\lambda_4 & \beta_1\lambda_5 & \beta_1\lambda_6 & \frac{\alpha_w \beta_1 P_0}{\Lambda\beta} \\ \beta_2\lambda_1 & (\beta_2 - \beta)\lambda_2 & \beta_2\lambda_3 & \beta_2\lambda_4 & \beta_2\lambda_5 & \beta_2\lambda_6 & \frac{\alpha_w \beta_2 P_0}{\Lambda\beta} \\ \beta_3\lambda_1 & \beta_3\lambda_2 & (\beta_3 - \beta)\lambda_3 & \beta_3\lambda_4 & \beta_3\lambda_5 & \beta_3\lambda_6 & \frac{\alpha_w \beta_3 P_0}{\Lambda\beta} \\ \beta_4\lambda_1 & \beta_4\lambda_2 & \beta_4\lambda_3 & (\beta_4 - \beta)\lambda_4 & \beta_4\lambda_5 & \beta_4\lambda_6 & \frac{\alpha_w \beta_4 P_0}{\Lambda\beta} \\ \beta_5\lambda_1 & \beta_5\lambda_2 & \beta_5\lambda_3 & \beta_5\lambda_4 & (\beta_5 - \beta)\lambda_5 & \beta_5\lambda_6 & \frac{\alpha_w \beta_5 P_0}{\Lambda\beta} \\ \beta_6\lambda_1 & \beta_6\lambda_2 & \beta_6\lambda_3 & \beta_6\lambda_4 & \beta_6\lambda_5 & (\beta_6 - \beta)\lambda_6 & \frac{\alpha_w \beta_6 P_0}{\Lambda\beta} \\ \frac{\Lambda\lambda_1}{C} & \frac{\Lambda\lambda_2}{C} & \frac{\Lambda\lambda_3}{C} & \frac{\Lambda\lambda_4}{C} & \frac{\Lambda\lambda_5}{C} & \frac{\Lambda\lambda_6}{C} & \frac{\alpha_w P_0}{C\beta} \end{bmatrix}$$

$$\delta CT(t) = \begin{bmatrix} \delta C_1(t) \\ \delta C_2(t) \\ \delta C_3(t) \\ \delta C_4(t) \\ \delta C_5(t) \\ \delta C_6(t) \\ \delta T_{ave} \end{bmatrix}$$

$$R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\dot{Q}_{SG}/c \end{bmatrix}$$

$$CT_0 = \frac{P_0}{\Lambda}\begin{bmatrix} \beta_1/\lambda_1 \\ \beta_2/\lambda_2 \\ \beta_3/\lambda_3 \\ \beta_4/\lambda_4 \\ \beta_5/\lambda_5 \\ \beta_6/\lambda_6 \\ 0 \end{bmatrix}$$

And

$$\frac{d(\delta CT)}{dt} = \Psi\, \delta CT + R$$

Is a linearized equation for a small perturbation around our fixed point. The matrix ψ is constant for a fixed point P_0 so we may use linear methods to study its eigenvalues and find the linear equation solution to observe the progress of the Perturbation in time.

$$\delta CT(t) = e^{\Psi t} \delta CT(0) + \Psi^{-1}(e^{\Psi t} - I)R$$

The system will be stable where the eigenvalues have no roots in the right half of the complex plane. The system oscillation will also depend on these roots. We will now go on to study the behavior of these roots as well as the implications of the various models:

- One Delayed Group
- Six Delayed Neutron Groups
- Full Reactor kinetics (See Note 1.)

<u>One Delayed Neutron Group</u>

Keep in mind this is a poor approximation group because it assumes a constant effective precursor decay constant. We do it only to simplify the computation as an example.

If we consider a one delayed group version of this problem, we obtain a two-by-two matrix as follows:

$$\Psi = \begin{bmatrix} 0 & \alpha_w P_0/\Lambda \\ \frac{\Lambda\lambda_{eff}}{c\beta} & \frac{\alpha_w P_0}{c\beta} \end{bmatrix} \qquad R = \begin{bmatrix} 0 \\ -Q_{SG}/c \end{bmatrix}$$

$$CT_0 = \frac{P}{\Lambda}\begin{bmatrix} \frac{\beta}{\lambda_{eff}} \\ 0 \end{bmatrix}$$

Upper left element is zero because the expression $\frac{\beta_1\lambda_1}{\beta} - \lambda_1 = 0$. We replaced β_1 with β and no other β_i exist.

Now compare the stability matrix with the model used for plotting reactor power and steam demand transients.

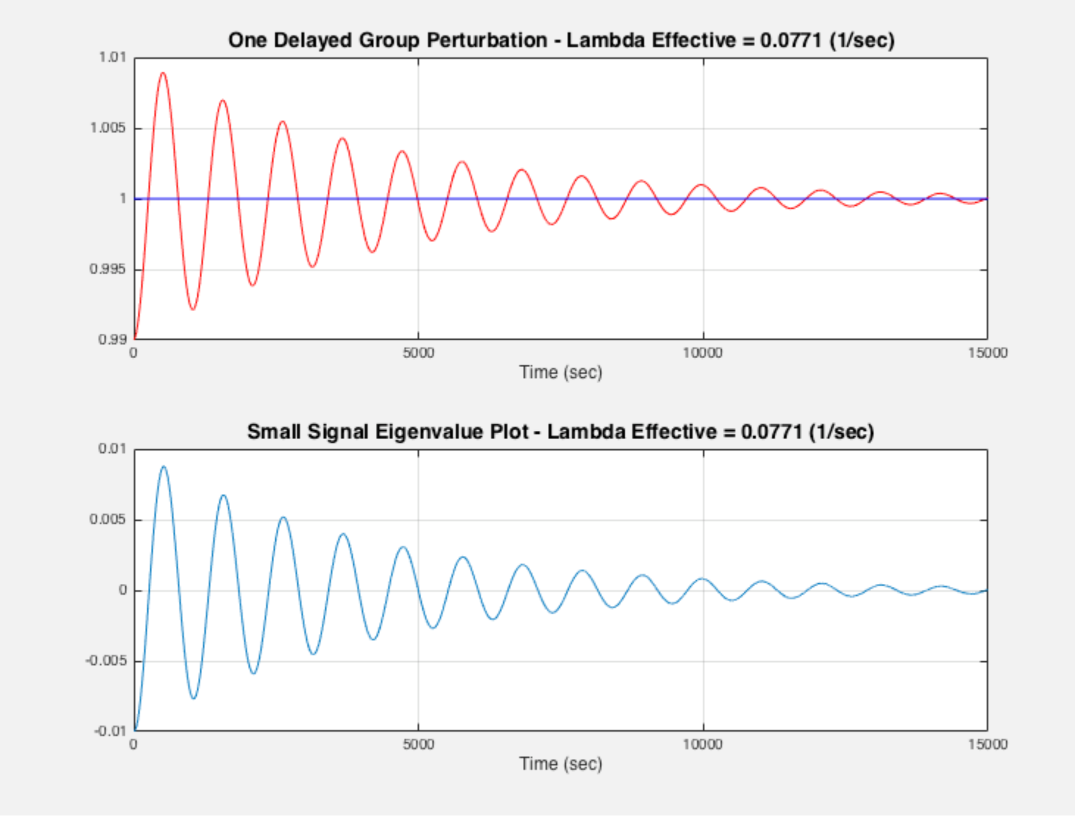
The matrix above has the following eigenvalues:

$$e_{\mp} = \frac{\alpha_w P_0}{2c\beta}\left[1 \mp (1 + \frac{4\lambda_{eff} c\beta}{\alpha_w P_0})^{1/2}\right]$$

Any value of $\alpha_w > 0$ will give an unstable root. Further if $\alpha_w < 0$ the system will have complex roots only if $\left|\frac{4\lambda_{eff}c\beta}{\alpha_w P_0}\right| > 1$. In that case the roots will appear as a conjugate pair. The complex roots imply a damped oscillation.

Suppose we establish initial conditions for 0.99% power but force a 1% steam flow. The system will allow power to rise to a steady state value of 1% but with extensive oscillations.

<u>Single Delayed Group Model Perturbation</u>
<u>1% Steam Flow</u>

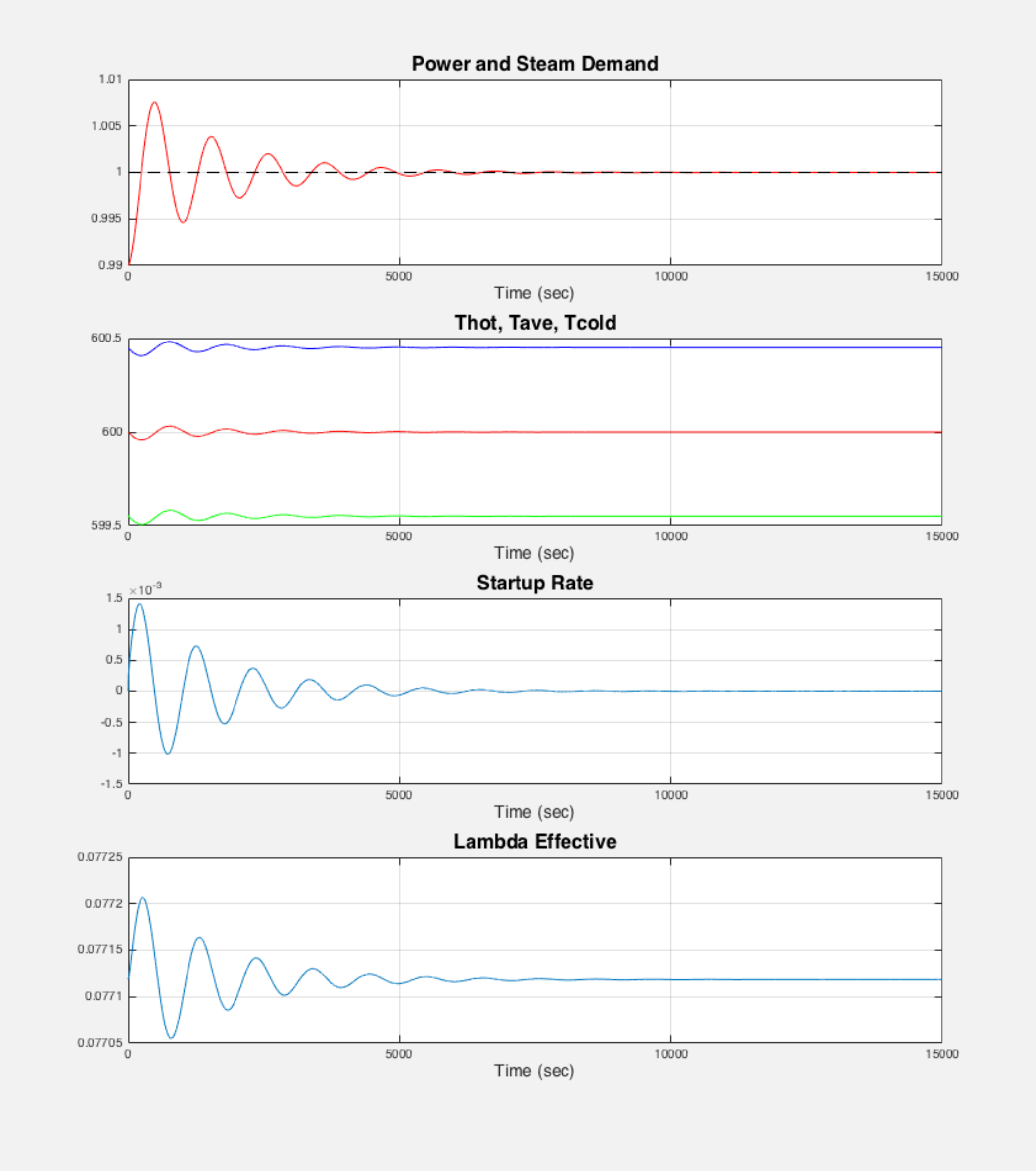


The ringing in these solutions is due to a small damping predicted by the single delayed group model. The real part of the two eigenvalues is -0 .00023 1/sec. So, the damping is of the form $e^{(-0.00023t)}$ at 1% power. This predicts a small power oscillation, which would continue for over four hours. <u>This model is incorrect</u>, power does not oscillate for this extend time. This largely due to ignoring $\frac{\dot{\lambda}_{eff}}{\lambda_{eff}}$ as we have discussed.

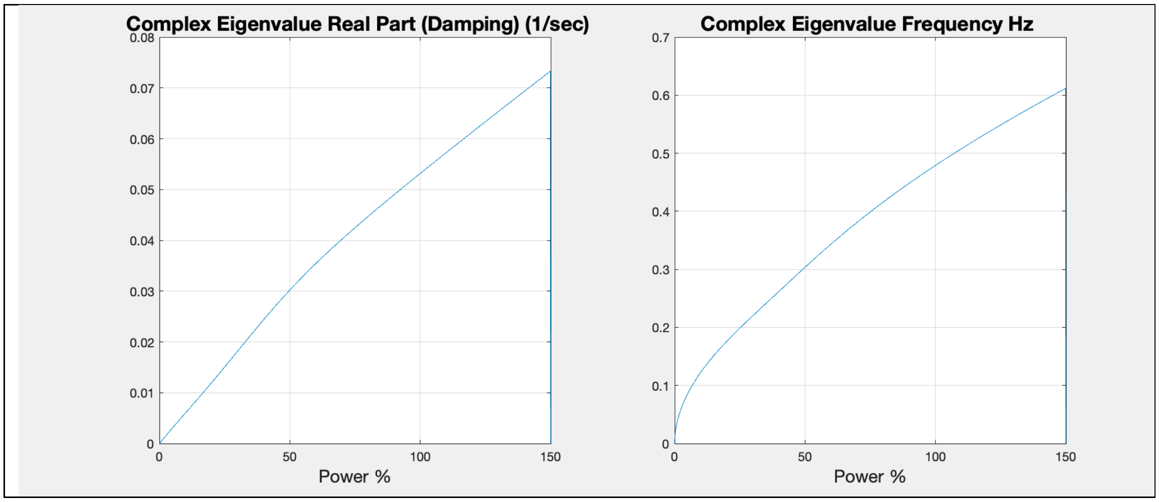
Six Delayed Neutron Groups

The six-group prompt jump assumption model is damped in about a third of the time of the one delayed group case.

Six Delayed Group Model Perturbation at 1% Steam Flow (Kinetics Solution)



The six-group matrix will have seven eigenvalues and for all power levels one pair of these eigenvalues will be complex. This differs from the one delayed group case.

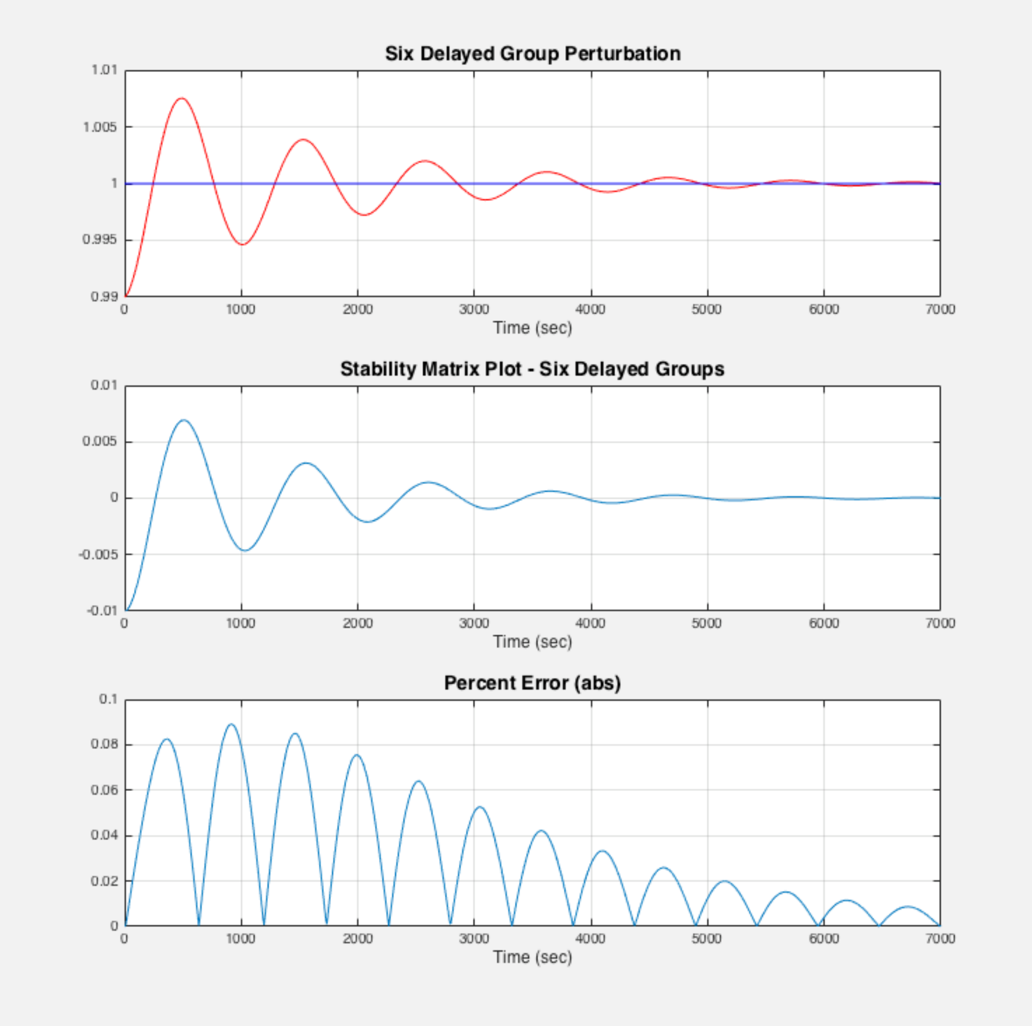


These plots tell us that at high power we may still see some overshoot but that the damping will be strong at those powers.

Six Delayed Group Solution and Stability Matrix Perturbation

1% Steam Flow

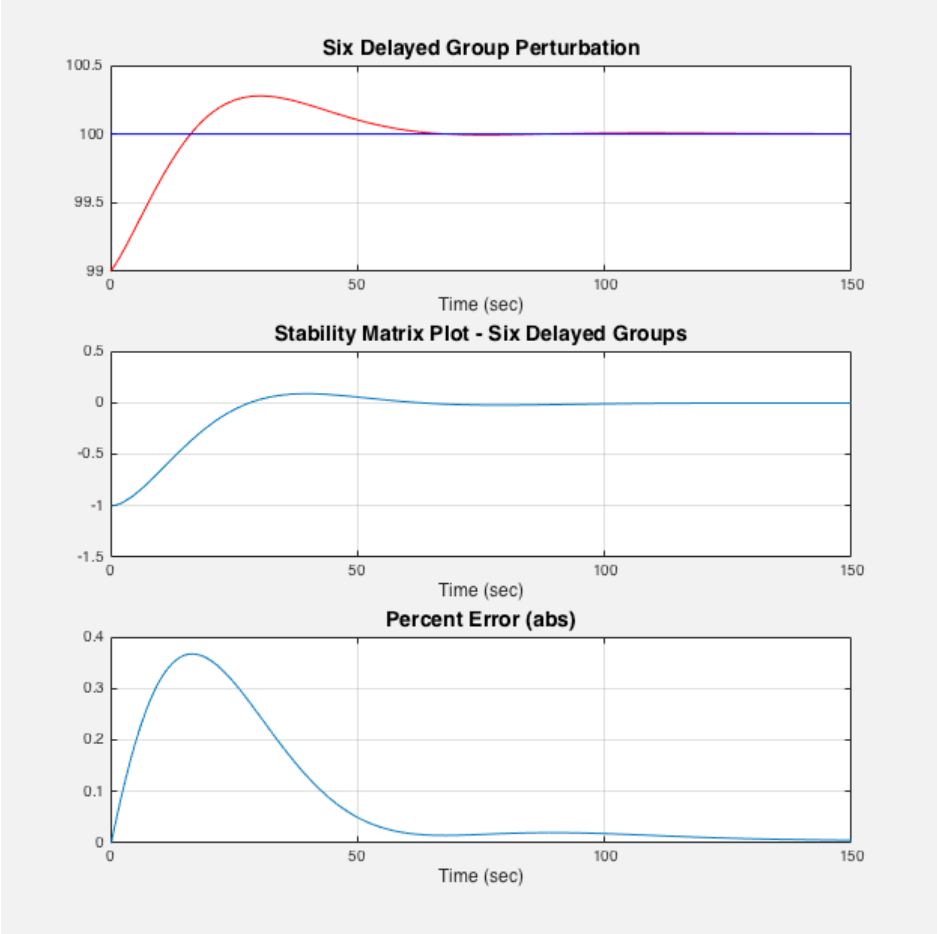
The first plot is a time domain plot for the transient using the solvers used previously in this paper. The second plot time domain solution of the stability matrix method of the system for the CT(t) vector of precursor concentrations. The subsequently power is computed from that vector.



It should be noted that the perturbation settles out to zero similar to the actual power transient. This plot shows the perturbation in power settling out to zero, leaving the power at its steady state value. The key here is that the frequency and duration of the transient is a close match to the full six-group solution.

This is the same case repeated for 100% steam flow.

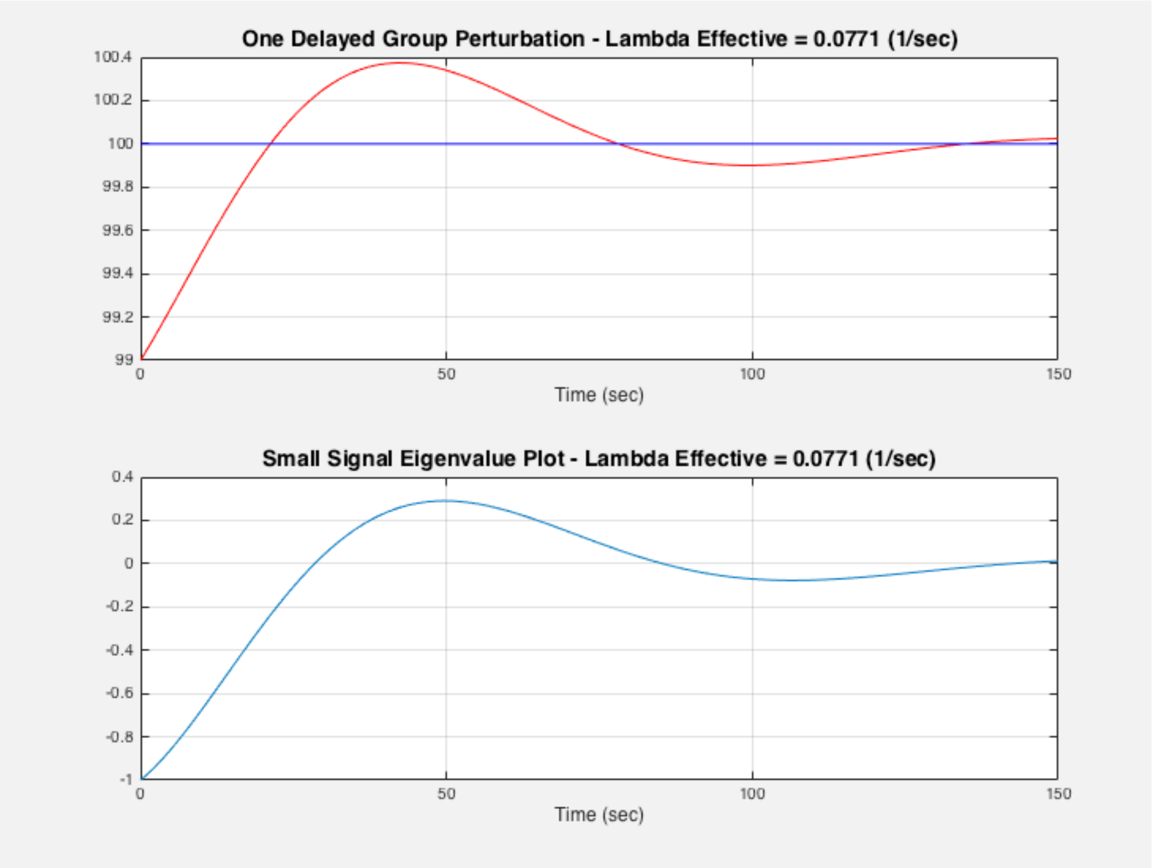
Six Delayed Group Solution and Stability Matrix Perturbation
100% Steam Flow



This is the full solution for the same 1% perturbation at 100% power. The matrix method (second plot) somewhat under predicts the overshoot and crosses zero later than the full solution shown here in the first plot. It is interesting to note that in the case of the 1% Steam Flow shown earlier, the agreement between the stability method and the full solution were nearly exact.

By contrast the single-delayed group method significantly over predicts the length of time for this transient to die out. The oscillation in that case continues to be observable for at least 150 sec as compared to about 70 sec in the six-delayed group case. The Stability Matrix approach for this case also exhibits the under predicted overshoot and later zero crossing as we saw for the six-group case.

One Delayed Group Model Perturbation at 100% Steam Flow



Conclusion

Use of the six-group Stability matrix may be helpful to obtain confidence in a model based on six delayed neutron groups and a single system T_{ave}. The one delayed group stability matrix is not recommended for this purpose.

Notes

1. No Prompt Jump Assumption.

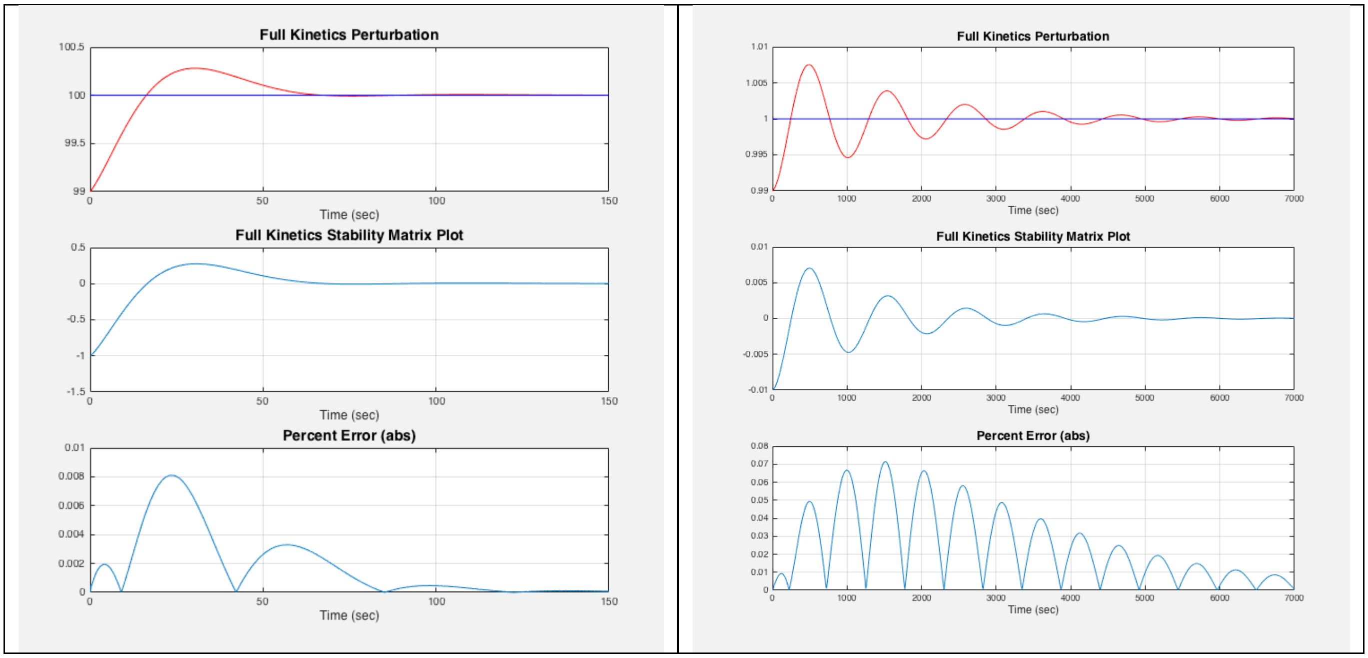
A full kinetics version of the stability matrix has the following form, Note we use S rather than Psi here. This S should not be confused with the

$$\Psi = \begin{bmatrix} -\frac{\beta}{\Lambda}+\frac{\alpha_F P_0}{\Lambda} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \frac{\alpha_w P_0}{\Lambda} \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_3}{\Lambda} & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & 0 \\ \frac{\beta_4}{\Lambda} & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & 0 \\ \frac{\beta_5}{\Lambda} & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & 0 \\ \frac{\beta_6}{\Lambda} & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & 0 \\ 1/C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$CT_0 = \frac{P_0}{\Lambda}\begin{bmatrix} \Lambda \\ \beta_1/\lambda_1 \\ \beta_2/\lambda_2 \\ \beta_3/\lambda_3 \\ \beta_4/\lambda_4 \\ \beta_5/\lambda_5 \\ \beta_6/\lambda_6 \\ 0 \end{bmatrix}$$

This is significantly better at 100% and slightly better at 1% power as compared to the six-group prompt jump version of the solution. This version also allows a fuel reactivity correction. The plot below has this correction set to zero to allow comparison with the previous plots. The error is computed as the absolute value of the difference divided by the full kinetics solution times 100%.

Full Kinetics Model Perturbation at 100% and 1% Steam Flow



Full Kinetics Perturbation

Full Kinetics Stability Matrix Plot

Percent Error (abs)

Full Kinetics Perturbation

Full Kinetics Stability Matrix Plot

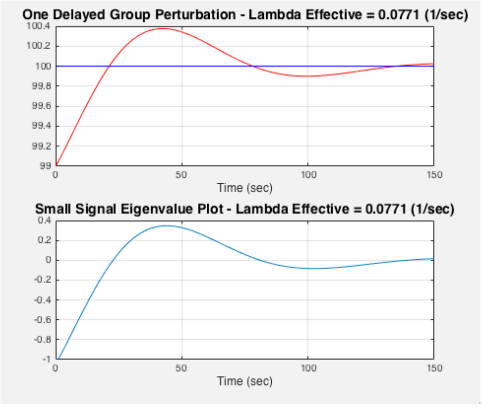
Percent Error (abs)

2. The analog one delayed group stability matrix is a three-dimensional matrix with a CT0 vector that contains variation in power, precursor concentration, and temperature. This system is as follows:

$$\Psi = \begin{bmatrix} -\frac{\beta}{\Lambda} + \frac{\alpha_F P_0}{\Lambda} & \lambda_{eff} & \frac{\alpha_W P_0}{\Lambda} \\ \frac{\beta}{\Lambda} & -\lambda_{eff} & 0 \\ 1/C & 0 & 0 \end{bmatrix}$$

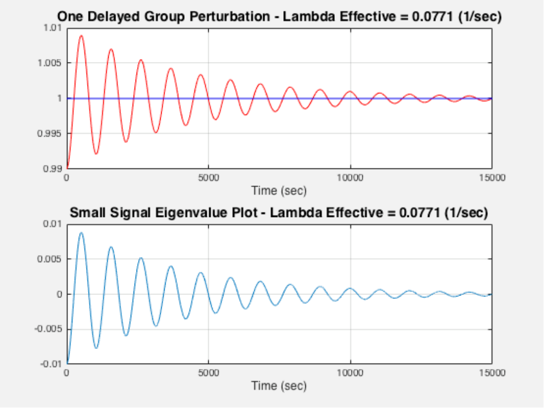
$$CT_0 = \frac{P_0}{\Lambda}\begin{bmatrix} \Lambda \\ \beta/\lambda_{eff} \\ 0 \end{bmatrix}$$

This version will be consistent with the one delayed group reactor kinetics equation. The excessive damping time will remain as discussed above. That is an inherent problem with the one delayed group.



One Delayed Group Perturbation - Lambda Effective = 0.0771 (1/sec)

Small Signal Eigenvalue Plot - Lambda Effective = 0.0771 (1/sec)



One Delayed Group Perturbation - Lambda Effective = 0.0771 (1/sec)

Small Signal Eigenvalue Plot - Lambda Effective = 0.0771 (1/sec)

Note 3. Transport Delay

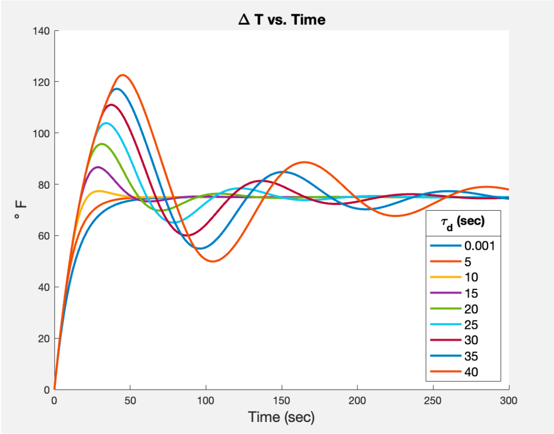
Real reactor plants pump fluid between the steam generator and reactor and the time to make a single loop will vary among the plants. Times may be as small as a few seconds in a small reactor system to a large fraction of a minute for reactors with low or no pumping power. For this discussion we will consider the impact on the hot and cold cycle temperatures given reactor and generator time constants. The delay times are half of the loop transport time. So $\tau_d^{Reactor} = \tau_d^{Generator} = \tau_d = \frac{Loop\ Transport\ Time}{2}$. We will not consider the impact on reactor kinetics here. We focus only on the fluid system transient as discussed in Section 2 of this paper.

The differential equations become the following.

$$\frac{dT_h}{dt} = \frac{T_c(t-\tau_d) - T_h(t)}{\tau_r} + \frac{P_r}{C_r}$$

$$\frac{dT_c}{dt} = \frac{T_h(t-\tau_d) - T_c}{\tau_s} - \frac{\dot{Q}_s}{C_s}$$

This system of equations does not allow normal solution because of the time delays. We will study this system using Laplace transforms but first, MATLAB does provide a means to directly integrate the equations with the time delays. Again using $\tau_r = \tau_s = 25\ sec$. We get the following on a step change in both reactor and generator power from zero to one hundred percent.



This set of graphs demonstrate that the impact of high loop transport times is to cause the system to have significant oscillation in its temperatures. At the same time, small delays cause the damping to be more rapid, note the case for a ten second loop transport time. However, beyond that value we see overshoot of increasing significance and a slowly damped oscillation.

The key to understanding this phenomenon is to consider this problem in the Laplace domain. We need a new fact about time delays and the Laplace transforms.

Assuming that τ_d is a constant, and that the transform of g(t) is G(s). then the transform g(t-τ_d) is $e^{-\tau_d s}$G(s). This may be demonstrated by going back and reviewing the integral definition of the transform and by making a time substitution in the integral.

With this, the Laplace transform of the differential equations become the following:

$$sT_h(s) = \frac{e^{-\tau_d s}T_c(s) - T_h(s)}{\tau_r} + \frac{P_r}{sC_r}$$

$$sT_c(s) = \frac{e^{-\tau_d s}T_h(s) - T_c(s)}{\tau_s} - \frac{\dot{Q}_s}{sC_s}$$

Put the temperature terms in vector form, $T = \begin{bmatrix} T_h(s) \\ T_c(s) \end{bmatrix}$ and rewrite our equations in the matrix form. Multiply by the time constants and collect all temperature terms on the left-hand side.

$$\begin{bmatrix} 1+\tau_r s & -e^{-\tau_d s} \\ -e^{-\tau_d s} & 1+\tau_s s \end{bmatrix} T = \frac{1}{s}\begin{bmatrix} \dfrac{\tau_r P_r}{C_r} \\ \dfrac{-\tau_s \dot{Q}_s}{C_s} \end{bmatrix}$$

Given this we can write a transfer function as follows:

$$T = \frac{\begin{bmatrix} 1+\tau_s s & e^{-\tau_d s} \\ e^{-\tau_d s} & 1+\tau_r s \end{bmatrix}}{(1+\tau_r s)(1+\tau_s s) - e^{-2\tau_d s}} \frac{1}{s}\begin{bmatrix} \dfrac{\tau_r P_r}{C_r} \\ \dfrac{-\tau_s \dot{Q}_s}{C_s} \end{bmatrix}$$

We have used the fact that the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-cb}$ and we have used the fact that squaring the exponential adds a two to the exponent.

So, our transfer function is:

$$\boxed{\mathrm{H} = \frac{\begin{bmatrix} 1+\tau_s s & -e^{-\tau_d s} \\ -e^{-\tau_d s} & 1+\tau_r s \end{bmatrix}}{(1+\tau_r s)(1+\tau_s s)-e^{-2\tau_d s}}}$$

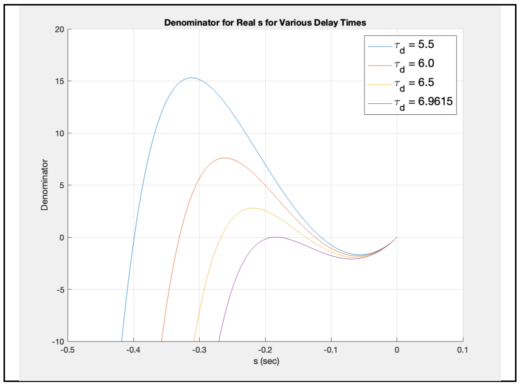
And

$$T = H \ \frac{1}{s}\begin{bmatrix} \frac{\tau_r P_r}{C_r} \\ \frac{-\tau_s \dot{Q}_s}{C_s} \end{bmatrix}$$

Studying the behavior of our system now starts by studying the roots of the denominator of H. Complex roots of this denominator will result in oscillation.

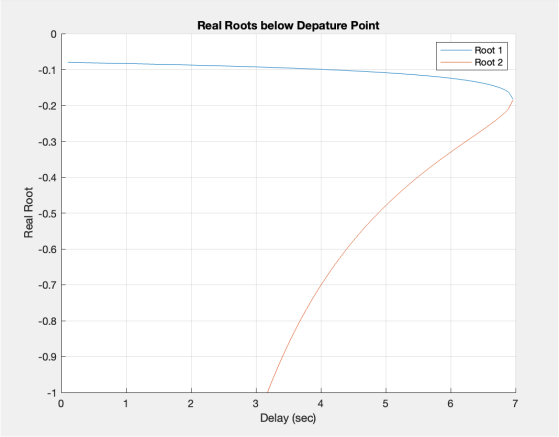
To begin, this denominator continues to provide a singularity at $s = 0.0$ for all values of the delay. This is because e^0 is one so the denominator becomes $(\tau_r + \tau_s)s + \tau_r\tau_s s^2$ and this will be zero if s is zero. The other root happens when $s = -(\tau_r + \tau_s)/\ \tau_r\tau_s = -1/\tau_{eff}.$

Using the values $\tau_r = \tau_s = 25\ sec$ we can numerically find roots that are <u>real</u> below a "Departure Value" at a delay time, $\tau_d = 6.9615$ sec. Below this Departure Value, the existence of the exponential contributes an additional real root. Plots of the denominator look as follows:



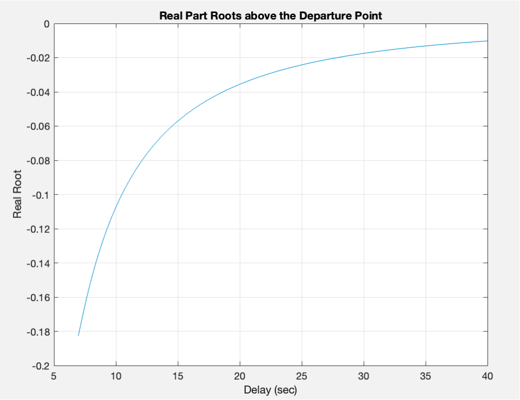
For all small positive values of τ_d the graphs have the same shape, crossing zero at two negative values of s leading to two exponential decay terms. These are the real roots. This is true until τ_d just is $\tau_d = 6.9615$ sec. At this point the hump no longer reaches the zero axis. From that point on the roots are complex leading to oscillation.

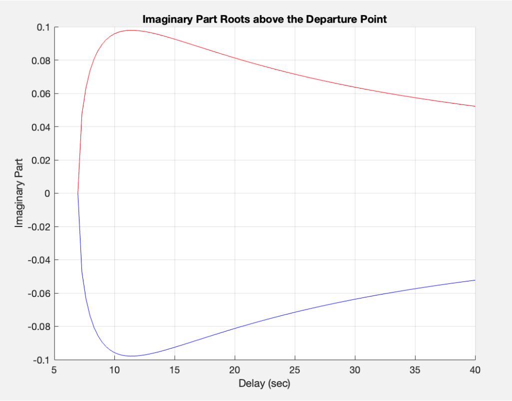
Here is a detailed plot of the real roots for the time delay below the Departure Point.



The root that is identified as "Root 2" will provide a term which will decay rapidly compared to the decay or the "Root 1" term. We expect the resulting plot to have a character driven by the later.

Once we reach the Departure Point, we will have complex roots leading to damped oscillation. The roots will have complex roots that appear in conjugate pairs.





Note that as the time delay increases beyond the Departure Point the magnitude of the real part of these eigenvalues becomes smaller. This means that the oscillations become less damped as the time delay is increased.

To understand the statement concerning conjugate pairs we will decompose the denominator into real and imaginary parts, both of which must be zero at a root.

$$(1 + \tau_r s)(1 + \tau_s s) - e^{-2\tau_d s}$$

Substitute $s = \sigma + j\omega$

$$(1 + \tau_r(\sigma + j\omega))(1 + \tau_s(\sigma + j\omega)) - e^{-2\tau_d(\sigma + j\omega)}$$

Multiply the left-hand side and write $e^{-2\tau_d(\sigma + j\omega)}$ as

$$e^{-2\tau_d\sigma}(\cos(2\tau_d\omega) + j\,sin(2\tau_d\omega)$$

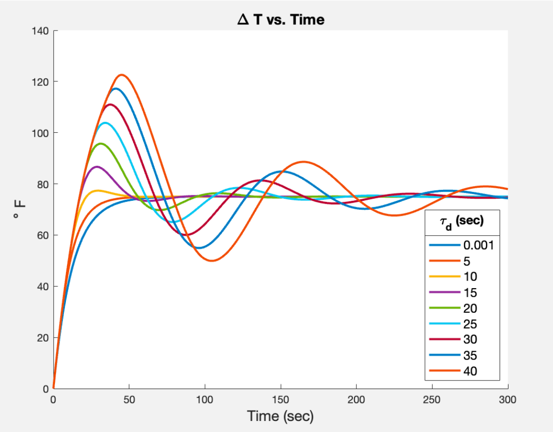
This results in the following two equations:

Real Part	$1 + \sigma(\tau_r + \tau_s) - \tau_r\tau_s(\sigma^2 - \omega^2) = e^{-2\tau_d\sigma}\cos(2\tau_d\omega)$
Imaginary Part	$\omega(\tau_r + \tau_s) = e^{-2\tau_d\sigma}\,sin(2\tau_d\omega)$

Examining the equations we see that if $\sigma + j\omega$ is a solution then, $\sigma - j\omega$ will also be a solution. We use the fact that the cosine function is even, and the sine function is odd.

We have focused this discussion only on the zero and other roots with the smallest magnitude real parts. These will dominate the solution. The two equations shown above will have a vast collection of possible solutions of the form $\sigma + j\omega$ due to the transcendental character of the sine and cosine functions. But, for reasonable values of the time delay, these roots will have a significantly larger damping constant and result in rapidly vanishing portions of the time domain solution.

Return to the time domain plot that we saw at the beginning of this discussion.



If we look at the case with a 40 second delay and measure the distance between the last two peaks in the red curve, we have roughly a 120 second period. The radial frequency at this point using our Imaginary Part above is roughly ω =0.0084 1/sec. The period would be $\frac{2\pi}{\omega}$ and this is 119.7 seconds. This demonstrates the relationship between the Laplace and time domain approaches.

Note 4. A Four Group Criticality Calculation

We solved a two-group example criticality calculation in this paper. It would be instructive to do a similar calculation assuming a four-energy group system. Real criticality calculations use far more than four energy groups, but the methods used in those cases are like what we will do here. We will continue to think of only a homogeneous one region reactor for this case. We will also approximate the flux as being zero at the boundaries of the cube.

Here is some typical data for four energy ranges:

	Group 1	Group 2	Group 3	Group 4
Energy Ranges	E > 0.821Mev	5.5 Kev<E<0.821Mev	0.625ev<E< 5.5Kev	E < 0.625ev
D	1.40 cm	0.9 cm	0.55 cm	0.30 cm
χ	0.98	0.02	0	0
$\nu\Sigma_f$	8.00x10^{-3} cm^{-1}	8.42x10^{-4} cm^{-1}	1.214 x10^{-2} cm^{-1}	1.32 x10^{-1} cm^{-1}
Σ_a	3.714x10^{-3} cm^{-1}	2.170x10^{-3} cm^{-1}	2.369 x10^{-2} cm^{-}	8.379 x10^{-2} cm^{-1}
$\Sigma_{s,out}$	0.0243 cm^{-1}	0.0628 cm^{-1}	0.570 cm^{-1}	0 cm^{-1}

In-Scattering Matrix

$$\Sigma_s^{\to g} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0243 & 0 & 0 & 0 \\ 0 & 0.0628 & 0 & 0 \\ 0 & 0 & 0.0570 & 1.3588 \end{bmatrix} \mathrm{cm}^{-1}$$

Note that the sum of the columns in this matrix with the diagonal elements removed is the scattering removal cross section for the associated group. We are using a simplified case where scattering moves us only one group down and self-scattering is ignored.

<u>These values have been manufactured to be reasonable but they do not represent any real reactor and should not be used beyond the context of this example.</u>

We can now write the diffusion equation for the multi group system as follows:

A

$$= \begin{bmatrix} {\Sigma_a}^1 + {\Sigma_{s,out}}^1 & -{\Sigma_s^{\to g}}_{1,2} & -{\Sigma_s^{\to g}}_{1,3} & -{\Sigma_s^{\to g}}_{1,4} \\ -{\Sigma_s^{\to g}}_{2,1} & {\Sigma_a}^2 + {\Sigma_{s,out}}^2 & -{\Sigma_s^{\to g}}_{2,3} & -{\Sigma_s^{\to g}}_{2,4} \\ -{\Sigma_s^{\to g}}_{3,2} & -{\Sigma_s^{\to g}}_{3,2} & {\Sigma_a}^3 + {\Sigma_{s,out}}^3 & -{\Sigma_s^{\to g}}_{3,4} \\ -{\Sigma_s^{\to g}}_{4,1} & -{\Sigma_s^{\to g}}_{4,2} & -{\Sigma_s^{\to g}}_{4,3} & {\Sigma_a}^4 + {\Sigma_{s,out}}^4 \end{bmatrix}$$

D

$$= \begin{bmatrix} D^1 & 0 & 0 & 0 \\ 0 & D^2 & 0 & 0 \\ 0 & 0 & D^3 & 0 \\ 0 & 0 & 0 & D^4 \end{bmatrix}$$

χ

$$= \begin{bmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \\ \chi^4 \end{bmatrix}$$

$$D^{4x4}\, B^2 \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix} = -A^{4x4} \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix} + \chi^{6x1}\, (\nu\Sigma_f{}^{6,1})^T \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix} = (\chi\, (\nu\Sigma_f)^T - A) \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix}$$

Note that the expression $\chi^{6x1}\, (\nu\Sigma_f{}^{6,1})^T$ creates a matrix, and the net result of its operation on the four-element flux is the scalar dot product of $\nu\Sigma_f$ times the flux vector. This scalar is then multiplied by the the vector χ.

The expression that we have here is a generalized eigen value problem:

$$\lambda N v \; = \; M v$$

Here $\lambda's$ are the eigenvalues, (in our case B^2) and eigenvectors are $v's$ are normalized flux vectors. For our case we will take only the smallest eigenvalue, as the buckling, and its associated eigenvector, as the flux vector. The buckling will be the smallest real positive eigenvalue for this equation. The values in the flux vector will give the relative amplitudes of the fluxes in each group. The overall values of these fluxes will then be determined by the power level. With only the thermal group being assumed to create fission here, the power will set the thermal group flux.

Many program environments can solve this directly. For example in MATLAB we can write [V,D] = eig(M,N) where V is a matrix of the right eigenvectors, and the matrix D has a diagonal consisting of the eigenvalues. We may also write this as [V,D] = eig(N\M).

The vector v (that corresponds with our buckling) when normalized will be the relative amplitudes of the group fluxes.

The side of the cube will then be $L\; = \sqrt{3}\; \pi/\sqrt{B^2}$. This the case because the buckling for a rectangular solid is:

$$B^2 = (\frac{\pi}{L_x})^2 + (\frac{\pi}{L_y})^2 + (\frac{\pi}{L_z})^2$$

This is true because we have the following:

$$\frac{\partial^2 F(x,y,z)}{\partial x^2} + \frac{\partial^2 F(x,y,z)}{\partial y^2} + \frac{\partial^2 F(x,y,z)}{\partial z^2} \; = \; -B^2 F(x,y,z)$$

This will be true only if $B^2 = \left[\frac{\pi m}{\hat{L}_x}\right]^2 + \left[\frac{\pi n}{\hat{L}_y}\right]^2 + \left[\frac{\pi p}{\hat{L}_z}\right]^2$ For the lowest mode we {m, n, p} all equal to one. For a cube the sides will all have equal length.

When we make this calculation, we get L = 77 cm. This number would be off because we are doing a four-group solution which fails to do justice to the problem, particularly in the resonance absorption region, we also have assumed that the flux goes to zero at the edges of the cube. This would be closer to correct a few centimeters outside the cube.

```matlab
% ComputeCriticalCubeFourGroup()
% William Locke
% October 4, 2025
%
% Make a four-energy group estimation of the size
% of a homogeneous cubic reactor with typical four
% group parameters that are only examples of values.
% They do not represent any real reactor.


function ComputeCriticalCubeFourGroup()
    D      = [1.40; 0.90; 0.55; 0.30];
    Chi    = [0.98; 0.02; 0.00; 0.00];
    nuSigmaf   = [8.00e-3; 8.42e-4; 1.214e-2; 1.328e-1];
    Sigma_a= [3.714e-3; 2.170e-3; 2.369e-2; 8.379e-2];
    % Scatter into groups, self-scattering is removed.
    InScattering = [ 0, 0,      0,      0;
               0.0243, 0, 0,      0;
               0,      0.0628, 0, 0;
               0,      0,      0.0570, 0];

    % This line sums the off-diagonal elements
    % one column at a time.  This gives the
    % scattering removal cross section for each
    % group.
    SigmaScatteringRemoval = sum(InScattering, 1).';
    Sigma_r = Sigma_a+SigmaScatteringRemoval;
    A = diag(Sigma_r) - InScattering ;

    % ---- Generalized eigen build ----
    M = Chi*(nuSigmaf.') -A;
    N = diag(D);
    B2Set = eig(N\M);
    % Select the real part of values
    % that are positive and nearly real.
    B2Set = B2Set(real(B2Set)>0);
    B2Set = B2Set(abs(imag(B2Set)) < 1.0e-10);
    B2Set = real(B2Set);
    % Now pick the minimum value for the remaining group.
    B2 = min(B2Set);
    L = pi*sqrt(3)/sqrt(B2);
    fprintf('Critical side L = %.3f cm\n', L);
end
```

Note 5. Reactivity

There are two views of the concept of reactivity, one which evaluates k_{eff} and defines reactivity as $\rho = 1 - 1/k_{eff}$. The other view is formed by the first order perturbation theory method used in deriving the reactor kinetics equation. For reactivities in normal operational ranges, these two methods produce equivalent results. For example, the tables below perturb the fission neutrons, the absorption cross section and the side length individually to measure the two reactivity computations.

$f\ with\ k_0 = 1$	$1.0005\upsilon\Sigma_f$	$1.0005\Sigma_a$	$1.0005L$
K Method	+33.73 pcm	-33.15 pcm	+28.07 pcm
Perturbation Method	33.75 pcm	-33.16 pcm	28.09 pcm

$f\ with\ k_0 = 1$	$1.01\upsilon\Sigma_f$	$1.01\Sigma_a$	$1.01L$
K Method	+670.39 pcm	-661.03 pcm	+552.89 pcm
Perturbation Method	674.91 pcm	-663.14 pcm	561.86 pcm

$f\ with\ k_0 = 1$	$0.95\upsilon\Sigma_f$	$0.95\Sigma_a$	$0.95L$
K Method	-3492.41 pcm	+3369.50 pcm	-3054.37 pcm
Perturbation Method	-3374.56 pcm	3315.69 pcm	-2809.28 pcm

K Method

D = Diffusion coefficients (diagonal matrix)
B_g^2 = geometric buckling for homogeneous shape
$R \ = \ diag(\Sigma_a + \Sigma_s^{out}) - S_{In-Scattering}^{Off\ Diagonal}$
A = Leakage + Absorption + Out-scatter
$A \ = \ DB_g^2 + R$
$F \ = \ \chi\left(\nu\Sigma_f\right)^T$
$F\varphi = kA\varphi \ \leftrightarrow A^{-1}F\varphi = \ k_{eff}\varphi$

Or

Perturbation method

$$\delta k/k \ = \frac{(\varphi^{\dagger})^T(\delta F - k_0\delta A)\varphi}{(\varphi^{\dagger})^T F\varphi}$$

The expression $\frac{\delta k}{k}$is very nearly the reactivity when k is nearly 1. $\rho = 1 - 1/k_{eff}$ so

$$\delta\rho = \delta k/k^2 \approx \delta k/k$$

Review of the derivation of the first order perturbation equation

1	$F\varphi = kA\varphi$
2	$F^T\varphi^{\dagger} = kA^T\varphi^{\dagger}$
3	$F \rightarrow F + \delta F, \qquad A \rightarrow A + \delta A,$
4	$\varphi \rightarrow \ \varphi + \delta\ \varphi, \qquad k \rightarrow \ k + \delta\ k$
5	$(F + \delta F)(\varphi + \delta\ \varphi) = (k + \delta\ k)(A + \delta A)(\varphi + \delta\ \varphi)$
6	$F\delta\ \varphi + \ \delta F\varphi = kA\delta\varphi + k\delta A\ \varphi + \ \delta\ kA\ \varphi$
7	$(\varphi^{\dagger})^T F = k(\varphi^{\dagger})^T A \ \rightarrow (\varphi^{\dagger})^T(F - kA) = 0$
8	$\delta\ k = \ (\varphi^{\dagger})^T(\delta F - k\delta A)\varphi/((\varphi^{\dagger})^T A\varphi)$
9	$A\varphi = 1/k\ F\ \varphi$
10	$\frac{\delta k}{k} = \frac{(\varphi^{\dagger})^T(\delta F - k\delta A)\varphi}{((\varphi^{\dagger})^T F\varphi)}$

So, this resulted in our equation for reactivity in terms of a fractional differential in k. It is this value that approximates $\rho = 1 - \frac{1}{k_{eff}}$ as we have shown above.

Note 6. Derivation of the Loop Temperature Related Equations

Derivation of the expression for e^{At}

$$A = \begin{bmatrix} -1/\tau_r & 1/\tau_r \\ 1/\tau_s & -1/\tau_s \end{bmatrix}$$

<table>
<tr><td></td><td colspan="3">$A = \begin{bmatrix} -1/\tau_r & 1/\tau_r \\ 1/\tau_s & -1/\tau_s \end{bmatrix}$</td></tr>
<tr><td>Let</td><td colspan="3">$\nu = \frac{\tau_s}{\tau_r}\ and\ \tau_{eff} = \frac{\tau_r \tau_s}{\tau_r + \tau_s}$</td></tr>
<tr><td rowspan="2">Then A, its eigenvalues and its modal matrix are:</td><td rowspan="2">$A = \frac{1}{\tau_{eff}(\nu + 1)} \begin{bmatrix} -\nu & \nu \\ 1 & -1 \end{bmatrix}$</td><td rowspan="2">$\{0, -1/\tau_{eff}\}$</td><td>$M = \begin{bmatrix} 1 & -\nu \\ 1 & 1 \end{bmatrix}$</td></tr>
<tr><td>$M^{-1} = \frac{\begin{bmatrix} 1 & \nu \\ -1 & 1 \end{bmatrix}}{\nu + 1}$</td></tr>
<tr><td></td><td colspan="3">$e^{At} = M \begin{bmatrix} 1 & 0 \\ 0 & e^{-t/\tau_{eff}} \end{bmatrix} M^{-1}$
$= \frac{1}{(\nu + 1)} \begin{bmatrix} \nu e^{-t/\tau_{eff}} + 1 & \nu - \nu e^{-t/\tau_{eff}} \\ 1 - e^{-t/\tau_{eff}} & \nu + e^{-t/\tau_{eff}} \end{bmatrix}$</td></tr>
<tr><td></td><td colspan="3">$e^{At} = \frac{1}{(\nu + 1)} \left(\begin{bmatrix} 1 & \nu \\ 1 & \nu \end{bmatrix} + e^{-t/\tau_{eff}} \begin{bmatrix} \nu & -\nu \\ -1 & 1 \end{bmatrix} \right)$</td></tr>
<tr><td>Add and subtract $\begin{bmatrix} \mu & -\mu \\ -1 & 1 \end{bmatrix}$</td><td colspan="3">$e^{At} = \frac{1}{(\mu + 1)} \left(\begin{bmatrix} 1 & \nu \\ 1 & \nu \end{bmatrix} + \begin{bmatrix} \nu & -\nu \\ -1 & 1 \end{bmatrix} - (1 - e^{-t/\tau_{eff}}) \begin{bmatrix} \nu & -\nu \\ -1 & 1 \end{bmatrix} \right)$</td></tr>
<tr><td></td><td colspan="3">$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{(1 - e^{-\frac{t}{\tau_{eff}}})}{(\nu + 1)} \begin{bmatrix} \nu & -\nu \\ -1 & 1 \end{bmatrix} = I + \tau_{eff}(1 - e^{-t/\tau_{eff}})A$</td></tr>
</table>

Derivation of the Loop Temperature Equation $e^{At}K$

We have the following equations:

$T = \begin{bmatrix} T_h \\ T_c \end{bmatrix}$	$A = \begin{bmatrix} -1/\tau_r & 1/\tau_r \\ 1/\tau_s & -1/\tau_s \end{bmatrix}$	$B = \begin{bmatrix} \frac{P_r}{C_r} \\ -\frac{\dot{Q}_s}{C_s} \end{bmatrix}$
	$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
$\frac{dT}{dt} = AT + B$	$T(t) = e^{At}T_0 + e^{At}\int_0^t e^{-At'} * B\, dt'$	

The matrix A has eigenvalues $\{0, \frac{1}{\tau_{eff}}\}$ and a modal matrix using the modal matrix and eigenvalues of A matrix we find the following:

$e^{At} = I + \tau_{eff}(1 - e^{\frac{t}{\tau_{eff}}})A$	$K = \int_0^t e^{-At'} * B\, dt'$
$K = t\tau_{eff}UB - \tau_{eff}^2\left(e^{\frac{t}{\tau_{eff}}} - 1\right)AB$	$U = \frac{I}{\tau_{eff}} + A$

The following identities hold:

(I+$\tau_{eff}A)A = 0$	$UB = 0\ if\ P_r = \dot{Q}_s$

$e^{At}K = t\tau_{eff}e^{At}UB - [I + \tau_{eff}(1 - e^{\frac{-t}{\tau_{eff}}})A]\tau_{eff}^2\left(e^{\frac{t}{\tau_{eff}}} - 1\right)AB$	
$e^{At}K = t\tau_{eff}e^{At}UB - [(I + \tau_{eff}A)A - \tau_{eff}e^{\frac{-t}{\tau_{eff}}})A^2B]\tau_{eff}^2\left(e^{\frac{t}{\tau_{eff}}} - 1\right)$	
$e^{At}K = t\tau_{eff}e^{At}UB - (-\tau_{eff}e^{\frac{-t}{\tau_{eff}}})A^2B]\tau_{eff}^2\left(e^{\frac{t}{\tau_{eff}}} - 1\right)$	(I+$\tau_{eff}A)A = 0$
$e^{At}K = t\tau_{eff}e^{At}UB - \tau_{eff}^2\left(1 - e^{\frac{-t}{\tau_{eff}}}\right)AB$	$\tau_{eff}A^2 = -A$
So $T(t) = e^{At}\left(\begin{bmatrix} T_h \\ T_c \end{bmatrix} + t\tau_{eff}UB\right)$ - $\tau_{eff}^2\left(1 - e^{\frac{-t}{\tau_{eff}}}\right)AB$	