University of Pittsburgh

ME/ENGR 2100 Fundamentals of Nuclear Engineering

Neutron Diffusion:

Multiplying Media

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Related Reading

- Chapter 5 of Duderstadt and Hamilton

OR

- Sections 5.1-5.7 Lamarsh and Baratta
- Sections 6.1-6.4 Lamarsh and Baratta

Learning Objectives

- Calculate the fundamental mode (scalar flux and multiplication factor) for a given power based on the one-group reactor equation for common geometrical configurations. Be able to find the reactor dimensions which will establish criticality for a given material composition.

Fission Neutron Source

- Previously we have not said much about the neutron source term.
  - In our derivations it has simply been represented as an unknown function.
- In a nuclear reactor our neutron source is due to neutrons produced during fission events.
- Let's see if we can derive an equation for the fission neutron source term, replacing the generic $s(\vec{r},t)$

Fission Neutron Source

- Derive an equation for the fission neutron source term [neutrons/second]
- Start with the fission reaction rate density:

$$\Sigma_f\left(\vec{r},t\right)\phi\left(\vec{r},t\right) \qquad \text{fissions / second}$$

  - This gives the instantaneous rate at which fissions are occurring.

Fission Neutron Source

- Start with the fission reaction rate density:

$$\Sigma_f(\vec{r},t)\,\phi(\vec{r},t)$$ fissions / second

- Each fission event releases ν [neutrons/fission] fission neutrons, on average.

- Fission neutron production rate density:

$$\nu\Sigma_f(\vec{r},t)\,\phi(\vec{r},t)$$ neutrons produced / second

Fission Neutron Source

$$\nu\Sigma_f\left(\vec{r},t\right)\phi\left(\vec{r},t\right)$$

- This is referred to as a multiplying source term (opposed to a fixed source term) because the magnitude of the source at every point depends on the flux at the point.
- Replacing our fixed-source gives us the diffusion equation in a multiplying medium

$$-D\frac{d^2}{d^2x}\phi\left(x\right)+\Sigma_a\phi\left(x\right)=\nu\Sigma_f\left(\vec{r},t\right)\phi\left(\vec{r},t\right)$$

Diffusion in Multiplying System

- The diffusion equation in a multiplying system allows us to describe the neutron population in a critical reactor
  - Equation as written on the previous slide assumes balance of production and loss
- Equation as written **only** has a solution for a critical mixture
  - Very unlikely to design a perfectly critical system on the first try
  - Not finding a solution does not give us any information about the criticality of the system
- Solution is to write problem as an eigenvalue problem
  - multiplication (k) is most common

Eigenvalue Problems

- In an eigenvalue problem we seek a nontrivial solution to some linear equation
  - Results are a set of eigenvalues and eigenfunctions if we are working with continuous operators
  - Eigenvalues and eigenvectors if we are talking about matrix equations
- Eigenvalue problems are prevalent in science and engineering
  - Vibrating strings / membranes
  - Structural mechanics
  - Molecular Orbitals

The k Eigenvalue

- To ensure we have a solution for any system configuration we imagine that the number of neutrons emitted per fission can be changed $\nu \rightarrow \frac{\nu}{k}$

- In this way <u>any</u> system can be made critical by choosing the appropriate value of k

$$-D\frac{d^2}{d^2x}\phi(x)+\Sigma_a\,\phi(x)=\frac{\nu}{k}\Sigma_f\,(\vec{r},t)\,\phi(\vec{r},t)$$

The k Eigenvalue

- There will be a largest value of k for which the scalar flux is nonnegative
  - If k=1 this implies the system is critical (time independent neutron balance)
  - If k<1 it implies the hypothetical number of neutrons per fission needs to increase
  - If k>1 it implies the hypothetical number of neutrons per fission needs to decrease

$$\nu \rightarrow \frac{\nu}{k}$$

$$k \begin{cases} > & 1 \quad \text{supercritical} \\ = & 1 \quad \text{critical} \\ < & 1 \quad \text{subcritical} \end{cases}$$

Diffusion in Multiplying Media

$$-D\frac{d^2}{dx^2}\phi(x)+\Sigma_a\,\phi(x)=\frac{\nu}{k}\Sigma_f\phi(x)$$

- Define material buckling:

$$B_m{}^2=\frac{\left(\frac{\nu}{k}\Sigma_f-\Sigma_a\right)}{D}$$

- We can then write the one-group reactor problem a $\frac{d^2}{dx^2}\phi(x)+B^2\,\phi(x)=0$

One-Group Reactor Equation

- We can also write this equation as

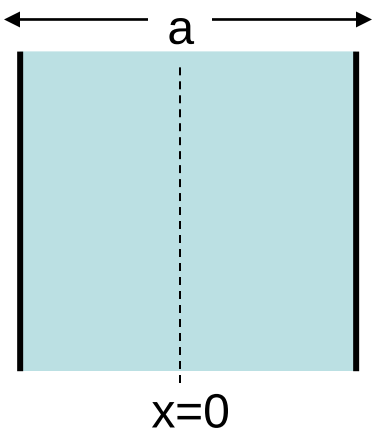
$$DB^2\phi(x)+\Sigma_a\,\phi(x)=\frac{\nu}{k}\Sigma_f\phi(x)$$

- Which can be solved for k, yielding

$$k=\frac{\nu\Sigma_f\phi(x)}{DB^2\phi(x)+\Sigma_a\,\phi(x)}=\frac{\nu\Sigma_f}{DB^2+\Sigma_a}$$

- Where B^2 is still unknown
  - Let's find the B^2

Bare Slab Reactor



- Infinite bare slab of thickness *a*

$$\frac{d^2}{dx^2}\phi(x)+B^2\,\phi(x)=0$$

- Zero flux boundaries

$$\phi\left(-\frac{a}{2}\right)=\phi\left(-\frac{a}{2}\right)=0$$

$$J(0)=0\rightarrow\frac{d}{dx}\varphi\Big|_{x=0}=0$$

- Due to symmetry no net flow through center of slab

Bare Slab Reactor

- General Solution

$$\phi(x) = c_1 \sin(Bx) + c_2 \cos(Bx)$$

- Imposing zero net current

$$\left.\frac{d}{dx}\phi(x)\right|_{x=0} = Bc_1 \cos(0) - B\,c_2 \sin(0) = 0$$

$$= Bc_1 \cos(0) = 0$$

$$c_1 = 0$$

$$\boxed{\phi(x) = c_2 \cos(Bx)}$$

Bare Slab Reactor

- Imposing zero flux boundaries

$$\phi\left(\frac{a}{2}\right) = c_2 \cos\left(\frac{Ba}{2}\right) = 0$$

  - Either $c_2 = 0$ (and flux = 0) or

$$\cos\left(\frac{Ba}{2}\right) = 0 \rightarrow B_n = \frac{n\pi}{a}, \quad n = 1,3,5,\ldots,\infty$$

- With this value of B^2 we find that

$$\phi_n(x) = c_n \cos\left(\frac{n\pi x}{a}\right)$$

Multiple Solutions

- The subscript *n* indicates there are many solutions which solve the one-group reactor equation, called harmonic modes

$$\phi_n(x) = c_n \cos\left(\frac{n\pi x}{a}\right)$$

- Really, any of the possible solutions are valid solutions

$$\phi(x) = \sum_{n=1,3,5,\ldots}^{\infty} c_n \cos\left(\frac{n\pi x}{a}\right) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{(2n-1)\pi x}{a}\right)$$

Fundamental Mode

- As n increase k decreases

$$k_n = \frac{\nu\Sigma_f}{DB_n^{\ 2} + \Sigma_a} = \frac{\nu\Sigma_f}{\frac{D\pi^2}{a^2}(2n-1)^2 + \Sigma_a}$$

- Higher order modes become increasingly subcritical (decreasing neutron population). If we wait long enough only 1st mode remains, called fundamental mode

$$k_1 = \frac{\nu\Sigma_f}{\frac{D\pi^2}{a^2} + \Sigma_a} = k_{\text{eff}}$$

- Reactor properties determined by the fundamental mode

Harmonic Modes

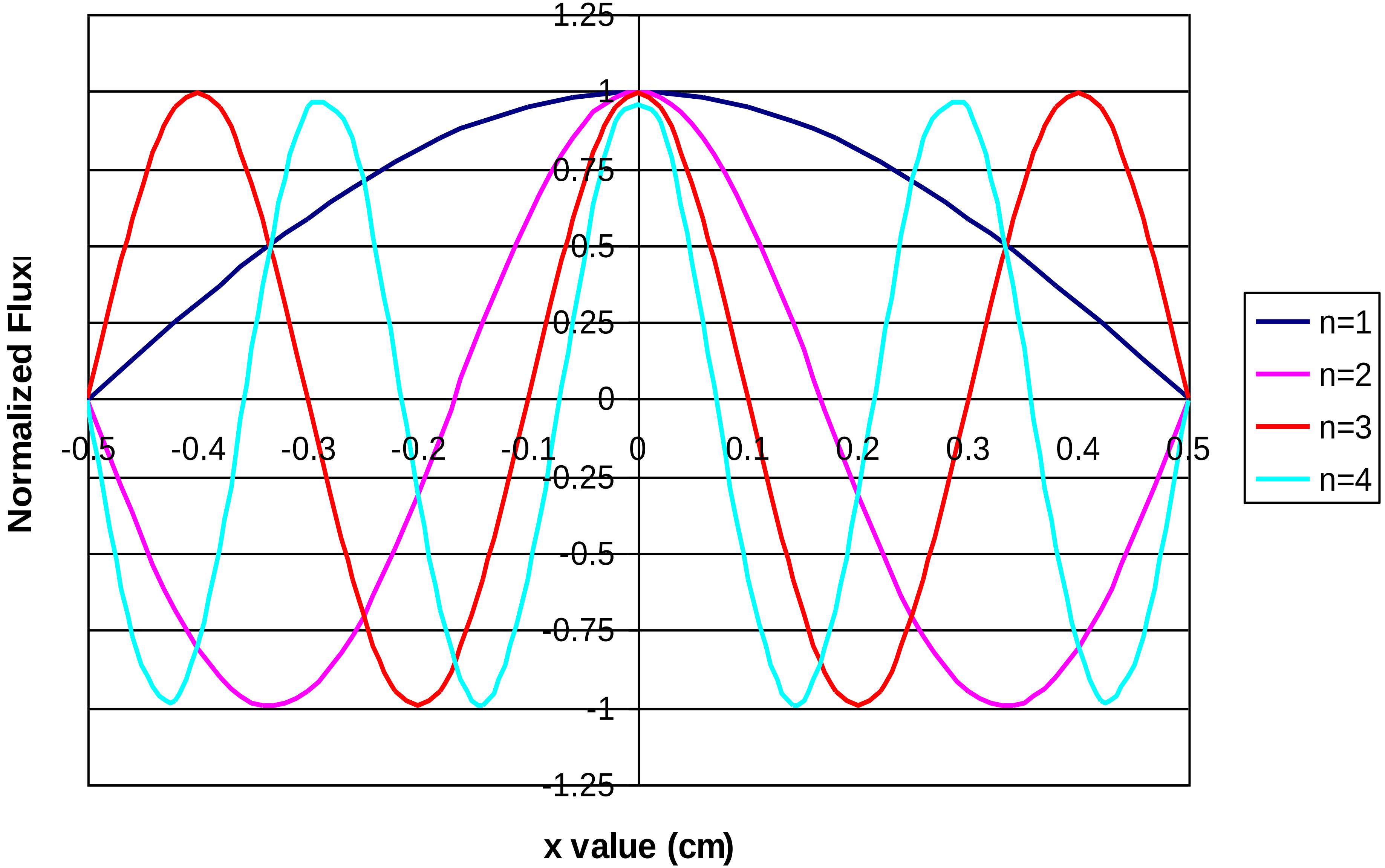
- Let’s consider the slab problem for a=1cm

$$\phi_n(x) = c_n \cos\left((2n-1)\pi x\right)$$

for the first few values of n (first few harmonic modes), where the flux is normalized by the as of yet undetermined c_n

- Plotting the scalar flux we see that only the fundamental mode is positive over the length of the slab
  - Confirms that it is the mode of interest since flux must be a positive quantity

First Few Flux Modes



Criticality Condition

Multiplication factor is given by

$$k = \frac{\nu\Sigma_f}{DB_g^{\ 2} + \Sigma_a}$$

where the buckling modes are given by

$$B_n = (2n-1)\frac{\pi}{a}, \quad n = 1,2,3,\ldots,\infty$$

and the geometric buckling is defined as

$$(B_1)^2 = \left(\frac{\pi}{a}\right)^2 = (B_g)^2$$

Criticality Condition

- Set k=1 and solve for geometric buckling

$$k = \frac{\nu\Sigma_f}{DB_g^{\;2} + \Sigma_a} \rightarrow DB_g^{\;2} + \Sigma_a = \nu\Sigma_f$$

$$B_g^{\;2} = \frac{\nu\Sigma_f - \Sigma_a}{D} = B_m^{\;2}$$

- In a critical system the geometric buckling is equal to the material buckling
  - To achieve criticality the system requires compatible materials and geometric configuration

Criticality Condition

- Geometric Buckling is a measure of the curvature of the flux in the reactor (measurement of the extent to which the flux curves/buckles)

$$\frac{d^2\phi}{dx^2} + B_1^2\phi = 0 \rightarrow B_1^2 = -\frac{1}{\phi}\frac{d^2\phi}{dx^2}$$

- Term comes from structural mechanics where the same equation can be used to describe the deformation of a beam under static load (buckling modes)

$$B_m^2 > B_g^2 \quad \text{supercritical}$$
$$B_m^2 = B_g^2 \quad \text{critical}$$
$$B_m^2 < B_g^2 \quad \text{subcritical}$$

Fundamental Mode

- We now know the flux and multiplication are described by the fundamental mode

$$\phi_1(x) = c_1 \cos\left(\frac{\pi x}{a}\right)$$

but we still need to find c_1

- To find a unique value of c_1 we can write it in terms of the current power level!

Power Calculation

- The power produced in the reactor is

$$P = E_R \Sigma_f \int_{-a/2}^{a/2} \phi(x)dx$$

$$= \frac{aE_R\Sigma_f c_1}{\pi} \sin\left(\frac{\pi x}{a}\right)\Bigg|_{-a/2}^{a/2} = \frac{2aE_R\Sigma_f c_1}{\pi}$$

$$c_1 = \frac{P\pi}{2aE_R\Sigma_f}$$

Power in watts/cm²

Macroscopic fission cross section

Slab width

Energy recoverable from fission (joules/fission)

$$\phi(x) = \frac{P\pi}{2aE_R\Sigma_f}\cos\left(\frac{\pi x}{a}\right)$$

General Geometries

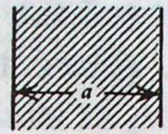
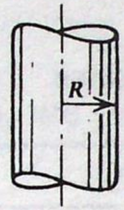
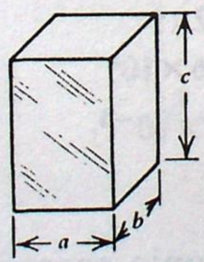
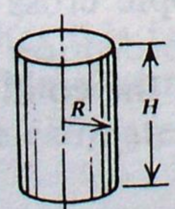
- We have solved the one-group reactor criticality problem for a slab by finding the geometric buckling and equating to the material buckling
- Can we do the same thing for other geometries?
  – Yes, in fact it is the exact same process

$$k = \frac{\nu\Sigma_f}{DB_g^{\ 2} + \Sigma_a}$$

Process for General Geometries

- Find geometric buckling
  - Solve differential equation for desired geometry OR
  - See Table 6.2 in L&B for common geometries
- Find the constant in terms of total power
- Find k by substituting the geometric buckling equation in
  - If searching for critical dimension then set material buckling to geometric buckling and solve for the desired dimension

TABLE 5-1 Geometric Bucklings and Critical Flux Profiles Characterizing Some Common Core Geometries

		Geometric Buckling B_g^2	Flux profile
Slab		$\left(\frac{\pi}{\tilde{a}}\right)^2$	$\cos\frac{\pi x}{\tilde{a}}$
Infinite Cylinder		$\left(\frac{\nu_0}{\tilde{R}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)$
Sphere		$\left(\frac{\pi}{\tilde{R}}\right)^2$	$r^{-1}\sin\left(\frac{\pi r}{\tilde{R}}\right)$
Rectangular Parallelepiped		$\left(\frac{\pi}{\tilde{a}}\right)^2+\left(\frac{\pi}{\tilde{b}}\right)^2+\left(\frac{\pi}{\tilde{c}}\right)^2$	$\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$
Finite Cylinder		$\left(\frac{\nu_0}{\tilde{R}}\right)^2+\left(\frac{\pi}{\tilde{H}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)\cos\left(\frac{\pi z}{\tilde{H}}\right)$

More Problems

- The majority of the diffusion material can be found in Lamarsh and Baratta
  - Example problems in the Sections 5.1-5.7 and 6.1-6.4 are good practice
  - Can find derivations for additional geometries
- Problems at the end of Chapters 5 and 6 cover material in the indicated sections