University of Pittsburgh

ME/ENGR 2100 Fundamentals of Nuclear Engineering

Thermal-Hydraulics Applied to Nuclear Reactors

Module 8.3

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Module Objectives

- Mathematical Concepts
- Heat Transfer Mechanisms
- Ideal Conduction

Mathematical Concepts

Gradient $\bar{\nabla}$

- The gradient of a scalar is a vector
- Points in the direction of maximum rate of change (units: quantity per unit length)
- Rectangular Coordinates

$$\bar{\nabla}T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

- Cylindrical Coordinates

$$\bar{\nabla}T = \frac{\partial T}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{u}_\theta + \frac{\partial T}{\partial z}\hat{u}_z$$

Laplacian ∇^2

- Divergence of a vector produces a scalar

$$\nabla \cdot \vec{T} = \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}$$

- Divergence of a Gradient is known as Laplacian $\bar{\nabla} \cdot \left(\bar{\nabla} T \right) = \nabla^2 T$

Rectangular Coordinates

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Cylindrical Coordinates

$$\nabla^2 T = \underbrace{\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r}}_{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplacian in 1-D ∇^2

- Generic form:

$$\nabla^2 f = \frac{1}{s\rho}\frac{\partial}{\partial s}\left(s\rho\frac{\partial f}{\partial s}\right)$$

- Can recover expression for rectangular, cylindrical, or spherical coordinates:

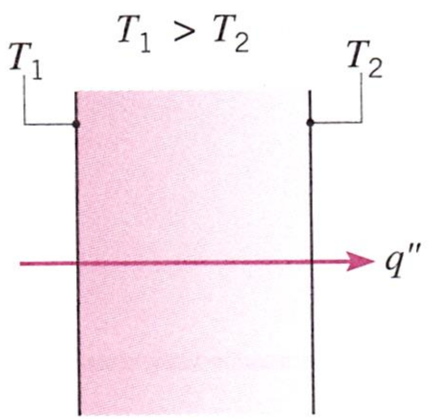
$$s = x \text{ and } \rho = 1/x \Rightarrow \text{rectangular}$$

$$s = r \text{ and } \rho = 1 \Rightarrow \text{cylindrical}$$

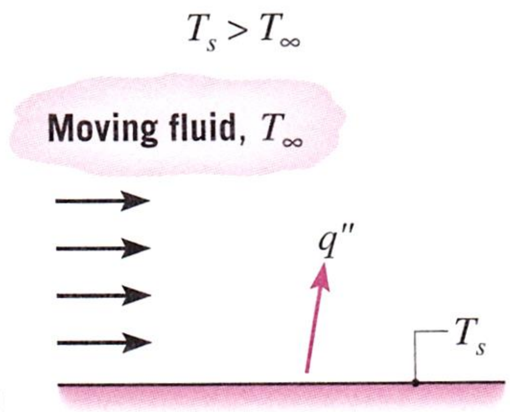
$$s = r \text{ and } \rho = r \Rightarrow \text{spherical}$$

Heat Transfer Mechanisms

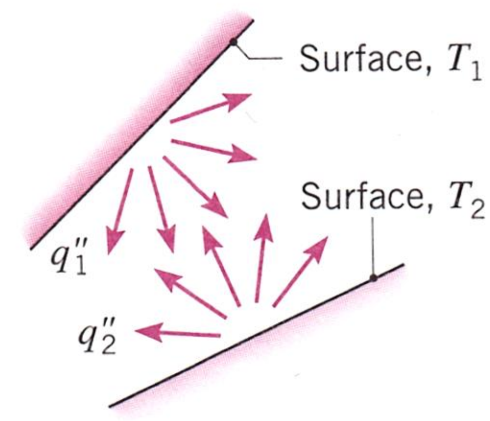
Heat Transfer Modes



Conduction



Convection



Thermal Radiation

Requires:
- Temperature Difference
- Medium to transfer heat

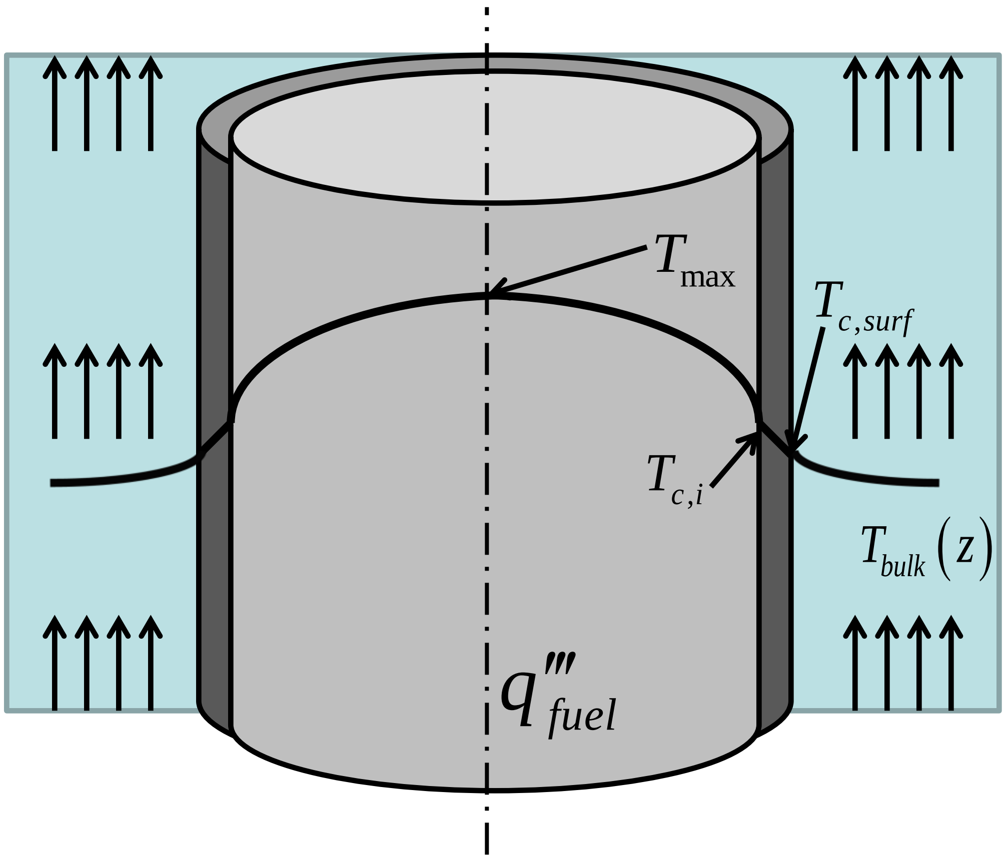
Requires:
- Temperature Difference
- Flowing or moving medium

Requires:
- Temperature Difference

Where does each mode apply to the analysis of a nuclear reactor?

Graphics from: “Incropera, F.P. and Dewitt, D.P., Fundamentals of Heat and Mass Transfer, 5th ed. , John Wiley & Sons, 2002

Nuclear Reactor Thermal Analysis



Idealized Fuel Rod

- Need to obtain temperature distribution within fuel element
  - Heat generated and transferred in fuel pellet
  - Heat transferred through cladding
  - Heat transferred to coolant
- Let's start with the fuel pellet

Derivation of the Conduction Equation

Heat Flux

- Flux is a basic concept without definition
- Numerous types
  - Mass flux
  - Momentum flux (viscous stress)
  - Neutron flux
  - Energy flux
- Can be a scalar, vector, or tensor all depending on the application
  - For heat flux, it is a vector normal to the surface of interest

Heat Flux

- Flux is defined, in 1-D, as:

$$Flux\left[\frac{quantity}{area-time}\right]=-Coefficient\left[\frac{area}{time}\right]\frac{d}{dn}(quantity\ density)\left[\frac{quantity}{volume}\right]$$

  - Neutron Current (Fick's Law)

$$J_n=-D\frac{d\phi}{dn}$$

  - Mass Diffusion

$$j_n=-D\frac{d}{dn}\rho$$

Heat Flux

- Heat Flux (Fourier's Law)
  - Quantity of interest is energy
  - In words:

$$Energy\ Flux\left[\frac{J}{m^2-s}\right] = -Coefficient\left[\frac{m^2}{s}\right]\frac{d}{dn}\left(Energy\ Density\right)\left[\frac{J}{m^3}\right]$$

  - In scalar form (1-D):

$$q_x'' = -\alpha\frac{d}{dx}E''' = -\frac{k}{\rho C_p}\frac{d}{dx}\left(\rho C_p T\right) = -k\frac{dT}{dx}$$

  - In vector form:

$$\vec{q''} = -k\left(\frac{dT}{dx}\hat{i} + \frac{dT}{dy}j + \frac{dT}{dz}k\right) = -k\nabla T\left(x, y, z\right)$$

  - Component wise (Cartesian coordinates)

$$q_x'' = -k\frac{dT}{dx}, q_y'' = -k\frac{dT}{dy}, q_z'' = -k\frac{dT}{dz}$$

Heat Flux

- Thermal diffusivity

$$\alpha = \frac{k}{\rho C_p}$$

  - Ratio of material's ability to conduct thermal energy relative to store thermal energy

– Thermal conductivity

$$k_x \equiv -\frac{q_x''}{\left(\partial T / \partial x\right)}$$

  - For an isotropic material:

$$k_x = k_y = k_z$$

Heat Conduction Equation

- Conservation of energy is a combination of both mechanical and thermal energy
- Can obtain the thermal energy equation by subtracting the mechanical energy equation
  - By doing this, what do we assume?
    - Laminar flow or solids
    - No viscous heating
    - Constant pressure and thermal conductivity

Heat Conduction Equation

$$\frac{d}{dt}\iiint_V (\rho u)\, dV + \iint_S \rho u \left(\vec{v} - \vec{v_s}\right) \bullet \vec{n} dS = \oiint_S \left[-\vec{q''} + \left(\overline{\overline{\tau}} - p\overline{\overline{I}}\right) \bullet \vec{v} \right] \bullet \vec{n} dS + \iiint_V \rho \vec{g} \bullet \vec{v} dV + \iiint_V q''' dV$$

Time rate of change of energy in the volume

Rate of energy loss by convection

Surface heat addition(s)

Viscous heating

Pressure work

Work due to body force

Volumetric heat generation

- Assumptions
  - Solid $\vec{v} = 0$
  - No viscous heating and constant pressure $\oiint_S \left[\left(\overline{\overline{\tau}} - p\overline{\overline{I}}\right) \bullet \vec{v} \right] \bullet \vec{n} dS = 0$
  - Control volume constant $\vec{v_s} = 0$

$$\frac{d}{dt}\iiint_V (\rho u)\, dV = \oiint_S -\vec{q''} \bullet \vec{n} dS + \iiint_V q''' dV$$

Heat Conduction Equation

$$\frac{d}{dt}\iiint_V (\rho u)\, dV = \iint_S -\overrightarrow{q''} \bullet \vec{n} dS + \iiint_V q''' dV$$

- From Gauss's Divergence Theorem

$$\iint_S -\overrightarrow{q''} \bullet \vec{n} dS = \iiint_V -\vec{\nabla} \bullet \overrightarrow{q''} dV$$

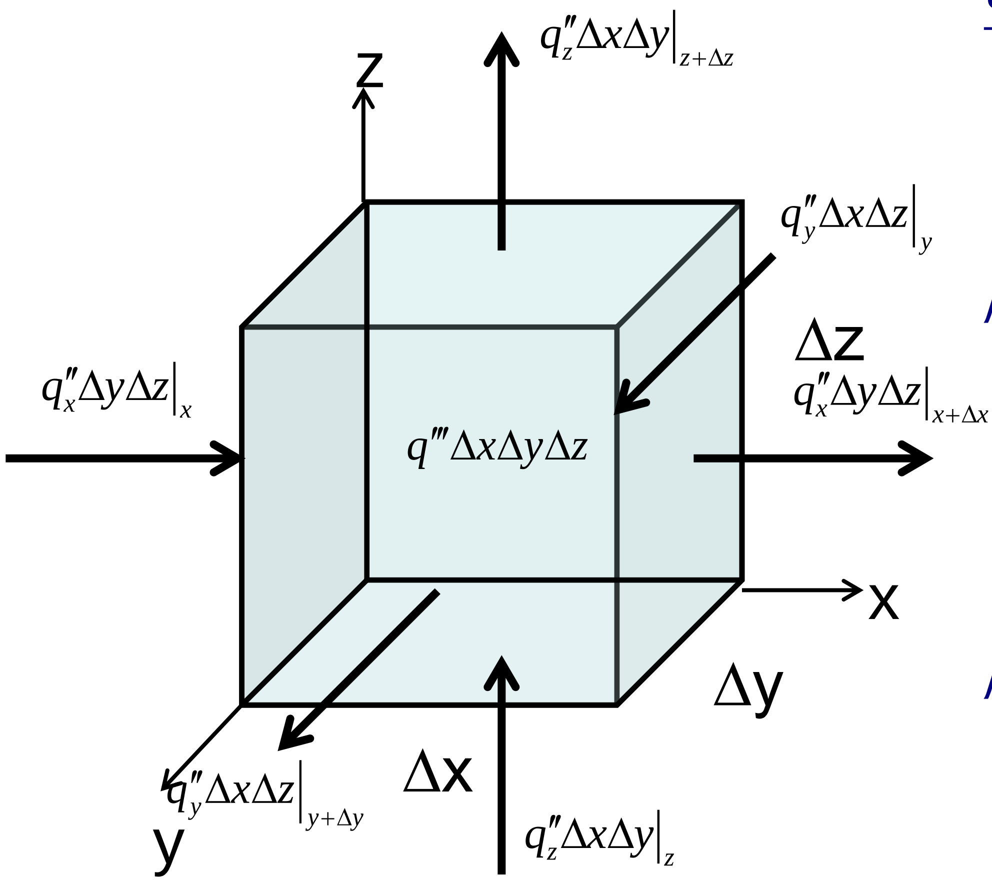
  - Means the volume integral of the divergence of the heat flux vector is equal to the total flux of the vector at the surface

$$\frac{d}{dt}\iiint_V (\rho u)\, dV = \iiint_V -\vec{\nabla} \bullet \overrightarrow{q''} dV + \iiint_V q''' dV$$

$$\frac{dE}{dt} = \frac{dmc_pT}{dt} = \frac{d\rho c_pT(Vol)}{dt} = -\vec{\nabla} \bullet \left(-k\vec{\nabla}T\right)Vol + q'''Vol$$

$$\rho c_p \frac{dT}{dt} = k\nabla^2 T + q'''$$

Conduction Equation



$$\frac{\partial E}{\partial t}=\frac{\partial m c_p T}{\partial t}=\frac{\partial \rho c_p T}{\partial t}(\Delta x\Delta y\Delta z)$$

$$\rho c_p \frac{\partial T}{\partial t}Vol=\begin{bmatrix} q_x''\Delta y\Delta z\big|_x - q_x''\Delta y\Delta z\big|_{x+\Delta x}+ \\ q_y''\Delta x\Delta z\big|_y - q_y''\Delta x\Delta z\big|_{y+\Delta y}+ \\ q_z''\Delta x\Delta y\big|_z - q_z''\Delta x\Delta y\big|_{z+\Delta z}+ \\ q'''\Delta x\Delta y\Delta z \end{bmatrix}$$

$$\rho c_p \frac{\partial T}{\partial t}=\begin{bmatrix} \dfrac{q_x''\big|_x - q_x''\big|_{x+\Delta x}}{\Delta x}+\dfrac{q_y''\big|_y - q_y''\big|_{y+\Delta y}}{\Delta y} \\ +\dfrac{q_z''\big|_z - q_z''\big|_{z+\Delta z}}{\Delta z}+q''' \end{bmatrix}$$

$$\rho c_p \frac{\partial T}{\partial t}=-\frac{\partial}{\partial x}(q_x'')-\frac{\partial}{\partial y}(q_y'')-\frac{\partial}{\partial z}(q_z'')+q'''$$

Derivatives

- Total Derivative:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial T}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial T}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + v_{o,x}\frac{\partial T}{\partial x} + v_{o,y}\frac{\partial T}{\partial y} + v_{o,z}\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \vec{v_o} \cdot \nabla T$$

  - where $\vec{v_o}$ is the velocity of the observer with respect to inertial coordinate system (Euler)

- Material Derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \frac{\partial T}{\partial t} + \left(\vec{v} - \vec{v_o}\right) \cdot \nabla T$$

  - where $\vec{v}$ is the velocity of the fluid relative to the observer

Heat Conduction Equation

- For a flowing liquid/metal:

$$\rho c_p \frac{DT}{Dt} = \nabla\left(k \nabla T\right) + q'''$$

- For a solid:

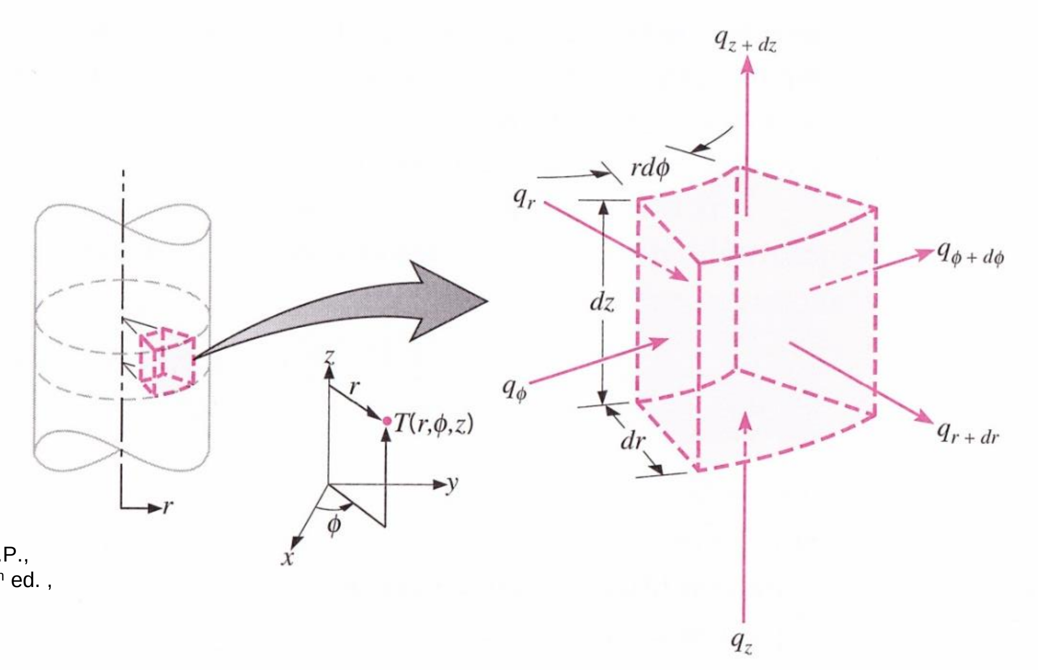
$$\rho c_p \frac{dT}{dt} = \rho c_p \frac{\partial T}{\partial t} = \nabla\left(k \nabla T\right) + q'''$$

- Scalar equation

Cylindrical Coordinates

$$k\vec{\nabla}T = k\frac{\partial T}{\partial r}\hat{i} + \frac{k}{r}\frac{\partial T}{\partial \phi}j + k\frac{\partial T}{\partial z}k$$

$$\nabla\cdot\left(\nabla kT\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$



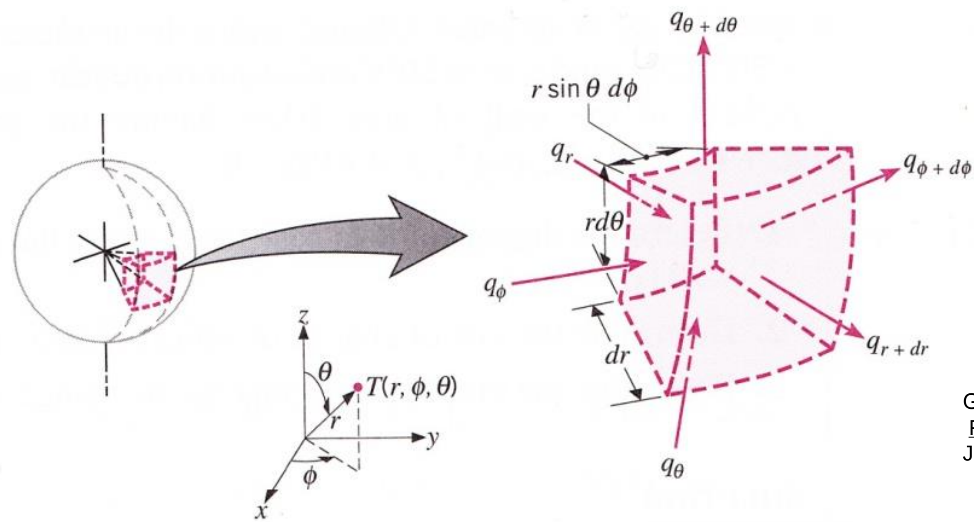
Graphics from: "Incropera, F.P. and Dewitt, D.P., Fundamentals of Heat and Mass Transfer, 5th ed. , John Wiley & Sons, 2002

Spherical Coordinates

- Can be derived in a similar manner.

$$k\vec{\nabla}T = k\frac{\partial T}{\partial r}\hat{i} + \frac{k}{r}\frac{\partial T}{\partial \theta}j + \frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}k$$

$$\nabla \cdot (\nabla kT) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)$$



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Forms of Conduction Equation

- For a solid with constant *k*:

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + q'''$$

  - Steady State
    - With heat generation → Poisson's Equation $k\nabla^2 T + q''' = 0$
    - Without heat generation → LaPlace's Equation $\nabla^2 T = 0$
  - Transient
    - Lumped Capacitance→ Newton's Equation $\frac{\partial T}{\partial t} = \frac{q'''}{\rho c_p}$
    - Without heat generation → Fourier's Equation

$$\frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \nabla^2 T$$

Initial & Boundary Conditions

- Heat conduction equation is 1st order in time
  - Requires one initial condition

$$T\left(\vec{r},t=0\right)=f\left(\vec{r}\right)$$

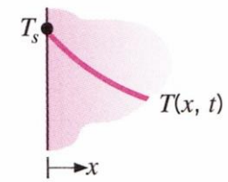
- Heat conduction equation is 2nd order in space
  - Requires two boundary conditions
    - 1st kind (Dirichlet) → Constant temperature
    - 2nd kind (Neumann) → Specified heat flux at surface
    - 3rd kind (Cauchy) → Convective boundary condition

$$-k\frac{\partial T}{\partial n}=h\left(T_{bulk}-T_{amb}\right)$$

Initial & Boundary Conditions

1. Constant surface temperature

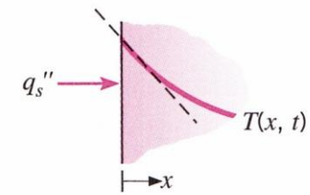
$$T(0, t) = T_s \qquad (2.24)$$



2. Constant surface heat flux

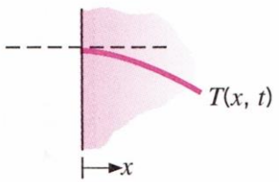
(a) Finite heat flux

$$-k \frac{\partial T}{\partial x}\bigg|_{x=0} = q_s'' \qquad (2.25)$$



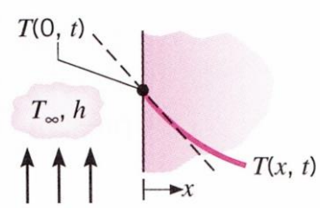
(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \qquad (2.26)$$



3. Convection surface condition

$$-k \frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \qquad (2.27)$$



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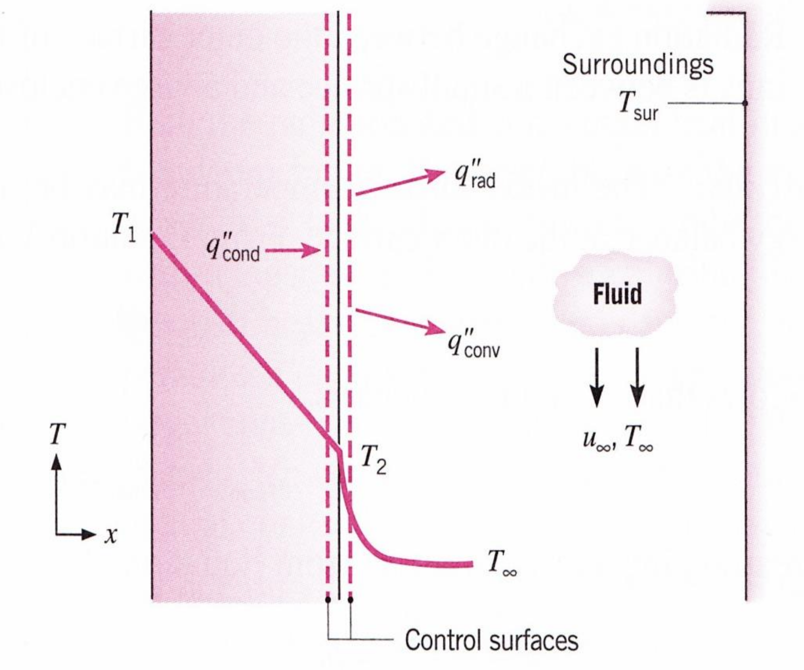
Interface Constraints

- At interfaces between different materials (i.e., fuel & cladding, pipe & insulation, etc.), must have:
  - Continuous temperature
    $$T_{left} = T_{right}$$
  - Continuous Heat Flux
    $$q''_{left} = q''_{right}$$
    $$-k_{left} \left.\frac{dT_{left}}{dr}\right|_{\text{interface}} = -k_{right} \left.\frac{dT_{right}}{dr}\right|_{\text{interface}}$$



Solving Heat Conduction Eqn.

- Typical assumptions:
  - 1-D (neglect axial and azimuthal conduction)
  - Steady-State (no storage of heat)
  - Volumetric Heat Source is produced uniformly throughout the fuel at a constant rate
- Treatment of Thermal Conductivity
  - Temperature dependent (fuel)
  - Temperature independent (cladding)
- How does one calculate the volumetric heat generation rate in nuclear fuel?

Example 1

- A copper rod is electrically heated such that its volumetric heat generation rate is 670 kW/m^3. The rod is 20cm in diameter and the surface temperature of the rod is 140° C. The conductivity of the cooper is constant at 390 W/m-K. Determine the steady-state temperature distribution in the rod and its maximum temperature. Neglect azimuthal and axial conduction.
- What happens if the rod was made of stainless steel instead (*k* = 18 W/m-K)

Example 2

- A cylindrical element is composed of two zones. The inner zone has a radius of 7/16”, a thermal conductivity of 103 BTU/hr-ft-F, and a volumetric heat generation rate of $1x10^6$ BTU/hr-ft^3.The outer zone has a radius of 5/8”, a thermal conductivity of 46.4 BTU/hr-ft-F, and no heat generation. The rod is cooled by convection with the bulk temperature at 400F and a heat transfer coefficient of 1000 BTU/hr-ft^2-F. Neglecting azimuthal and axial conduction, determine the steady state temperature distribution in the element.

Give this example a try. We will cover it in the review session.