University of Pittsburgh

ME/ENGR 2100 Fundamentals of Nuclear Engineering

Fission Reactor Basics:

Neutron Multiplication

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Relevant Reading Assignments

- Sections 6.5 to 6.8 of "*Introduction to Nuclear Engineering*" by Lamarsh and Baratta, 3rd Edition.
- Chapter 3 of "*Nuclear Reactor Analysis*" by Duderstadt and Hamilton
- Page 100-120 of "*Nuclear Engineering: Theory and Technology of Commercial Nuclear Power*" by Knief, 2nd Edition.

Relevant Reading Assignments

- "Secrecy, simultaneous discovery, and the theory of nuclear reactors" by Spencer Weart. American Journal of Physics, Vol. 45(11). November 1977

Learning Objectives

- Differentiate among critical, supercritical, and subcritical conditions in a reactor
- Identify the terms in the four and six factor formulas
- Explain the principle of neutron moderation by light nuclei and the importance to thermal reactors
- Understand the impact of heterogeneity on neutron balance
- Differentiate between the infinite (k_∞) and effective (keff or k) multiplication factors

Differentiate among critical, supercritical, and subcritical conditions in a reactor

Neutron Economy

- Nuclear reactor core design focuses on the neutron economy within a reactor during its operating lifetime
- A successful reactor design must
  - Produce enough excess neutrons to keep the chain reaction going
  - Limit the number of excess neutrons so that the reaction does not become uncontrolled
  - Consider thermal and material limits as well!
- Nuclear designers balance neutron sources (fuel) with neutron absorbers and leakage, the rate at which neutrons escape from the core.

Neutron Balance

$$\begin{bmatrix} \text{Rate of Increase} \\ \text{in Number} \\ \text{of Neutrons} \end{bmatrix} = \begin{bmatrix} \text{Rate of} \\ \text{Production} \\ \text{of Neutrons} \end{bmatrix} - \begin{bmatrix} \text{Rate of} \\ \text{Absorption} \\ \text{of Neutrons} \end{bmatrix} - \begin{bmatrix} \text{Rate of} \\ \text{Leakage} \\ \text{of Neutrons} \end{bmatrix}$$

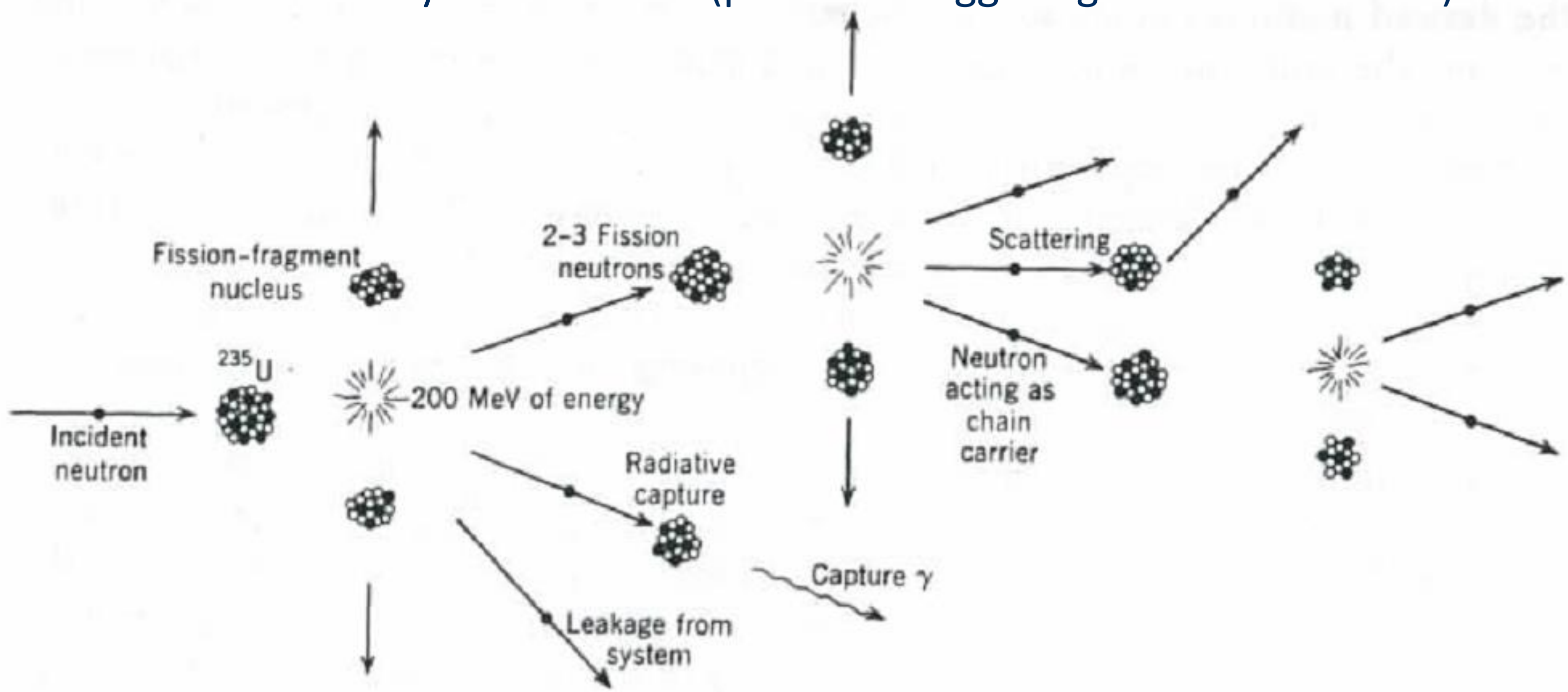
Accumulation = Production − Absorption − Leakage

If Accumulation:

= 0	Critical	Steady State	Static
> 0	Supercritical	Increasing	Kinetic/ Dynamic
< 0	Subcritical	Decreasing	Kinetic/ Dynamic

Neutron Life-Cycle

- Birth: Neutrons are born during fission events
- Lifetime: The lifetime of the neutron is the time between its birth and death. During this time the neutron potentially undergoes many scattering reactions off of host nuclei in the system
- Death: Neutron death occurs when the neutron leaks from the system or is absorbed by a host nuclei (potential triggering a fission reaction).



Life-Cycle Definition of k

- Accumulation boiled down to single number by defining multiplication factor, k

$$\boxed{k = \frac{\text{Number of neutrons in one generation}}{\text{Number of neutrons in preceding generation}}}$$

- Measures how many (average) neutrons are produced by each neutron born
- Characterizes the chain-reaction
  - Each neutron born must itself create at least 1 more neutron before being absorbed/leaking to sustain reaction

Criticality

- Critical: Reactor is static. The number of neutrons from generation to generation does not change
- Subcritical: Number of neutrons from generation to generation decreases, the reaction eventually dies out
- Supercritical: Number of neutrons from generation to generation increases without bound

$k<1$	subcritical
$k=1$	critical
$k>1$	supercritical

Neutron Balance Definition of k

$$\begin{bmatrix}\text{Rate of Increase} \\ \text{in Number} \\ \text{of Neutrons}\end{bmatrix} = \begin{bmatrix}\text{Rate of} \\ \text{Production} \\ \text{of Neutrons}\end{bmatrix} - \begin{bmatrix}\text{Rate of} \\ \text{Absorption} \\ \text{of Neutrons}\end{bmatrix} - \begin{bmatrix}\text{Rate of} \\ \text{Leakage} \\ \text{of Neutrons}\end{bmatrix}$$

$$\frac{dN}{dt} = \text{Production rate - (Absorption+Leakage) rate } = P(t) - L(t)$$

$$\boxed{k = \frac{\text{Rate of neutron production}}{\text{Rate of neutron loss}} = \frac{P(t)}{L(t)}}$$

Simple Chain-Reaction Kinetics

$$\frac{dN}{dt} = P(t) - L(t)$$ Change in neutron count = production rate – loss rate

$$l = \frac{N(t)}{L(t)}$$ Neutron lifetime, N(t) is total neutron population at time t

$$\frac{dN}{dt} = P(t) - L(t) = \left[\frac{P(t)}{L(t)} - 1\right] L(t) = (k-1)L(t)$$

Rewrite balance in terms of k and l

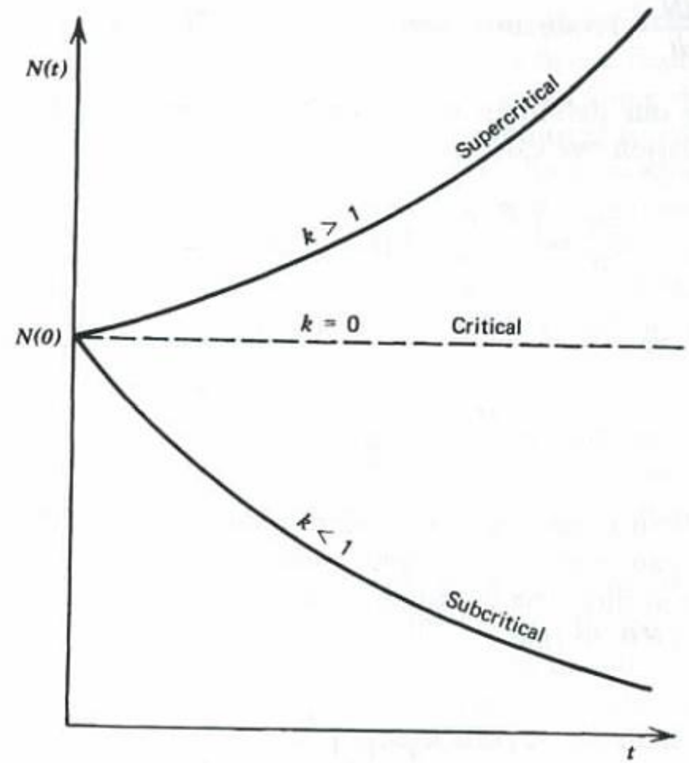
$$\boxed{\frac{dN}{dt} = \frac{(k-1)}{l} N(t)}$$

Simple Chain-Reaction Kinetics

- Solve simple differential equation to find

$$\boxed{N(t) = N_0 \exp\left[\frac{(k-1)}{l}t\right]}$$

- k is important in determining time-behavior of a reactor



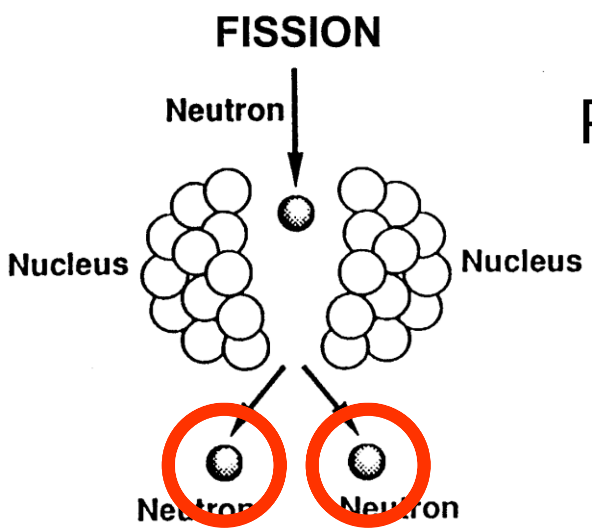
Determining k

- Finding k is crucial in reactor design
- Today the determination of k is done using mathematical theories and computer hardware not available to the first reactor designers
- Original theories primarily based on physical intuition and written in terms of experimentally measurable quantities
- These theories distinguish between infinite (easier to quantify) and finite systems (practical)

Infinite System

- Characteristics
  - No Outer Boundary $\leftrightarrow$ No Neutron Leakage
  - Abstraction to Simplify Calculations

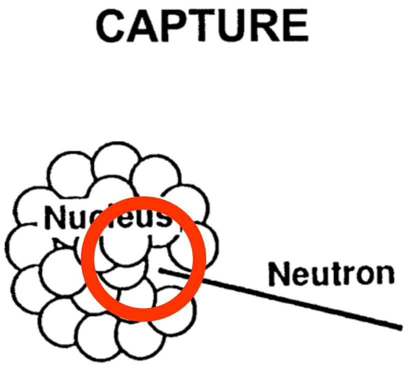
Nuclear Reactions

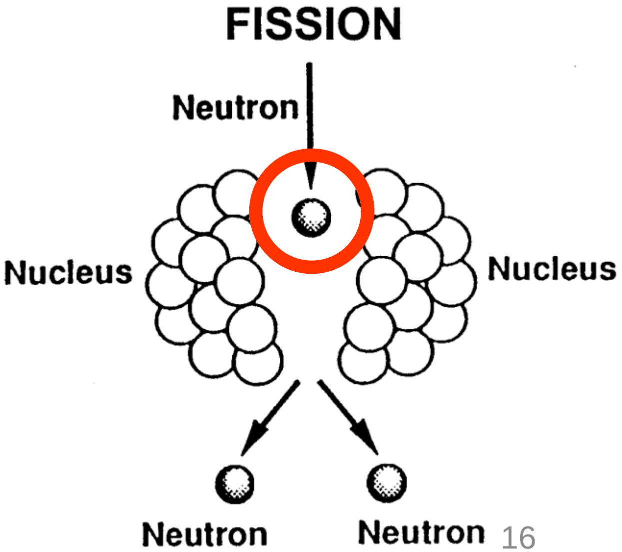


Production = Fission Rate ($\Sigma_f \Phi$)

× Neutrons produced per fission (ν)

Destruction Rate = Absorption Rate ($\Sigma_a \Phi$)





Infinite System

- Neutron Balance
  - Production Rate $\leftrightarrow$ Absorption Rate

$$\nu\Sigma_f\Phi \qquad\qquad \Sigma_a\Phi$$

  - (Infinite) Multiplication Factor

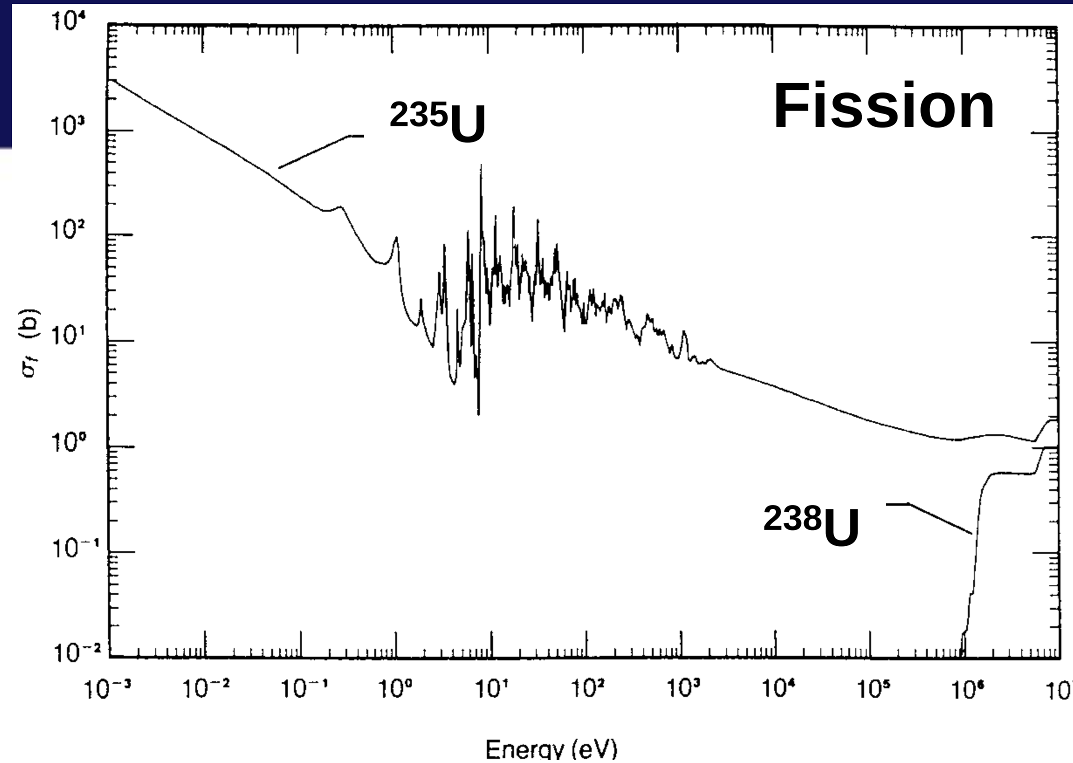
$$\boxed{k_\infty = \frac{\text{Production Rate}}{\text{Absorption Rate}} = \frac{\nu\Sigma_f}{\Sigma_a}}$$

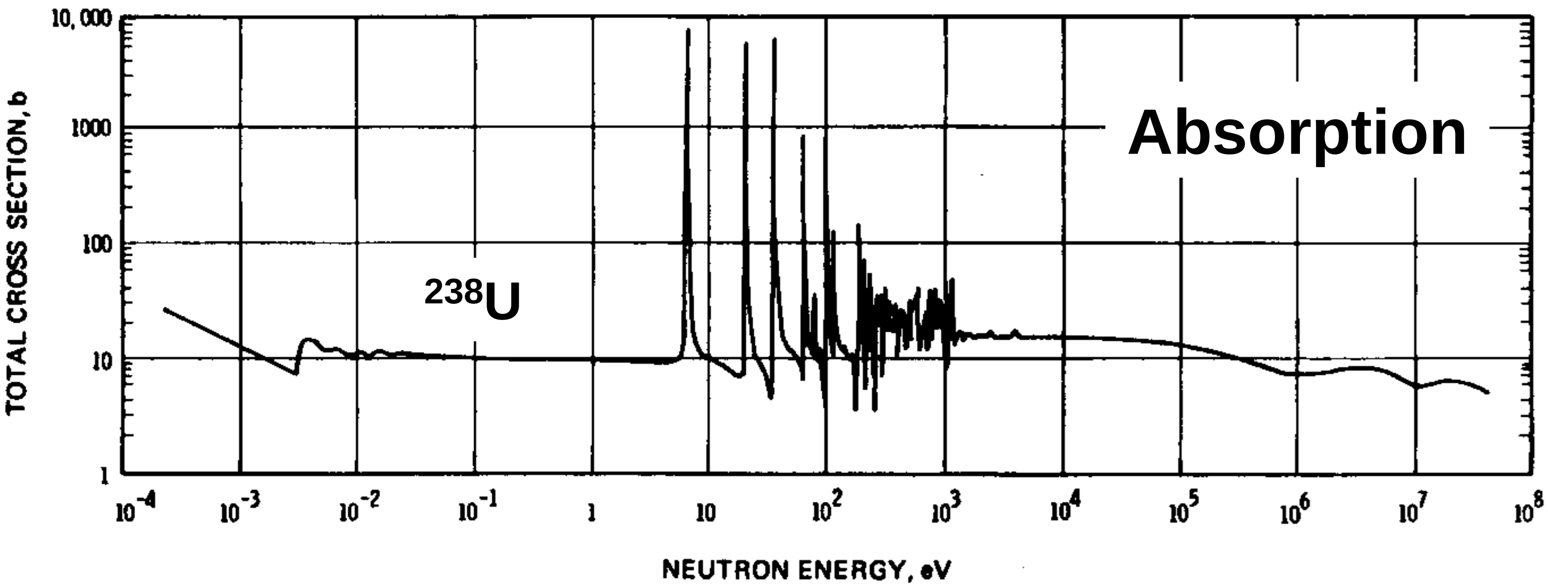
    - What about the neutron energy dependence?

Infinite System

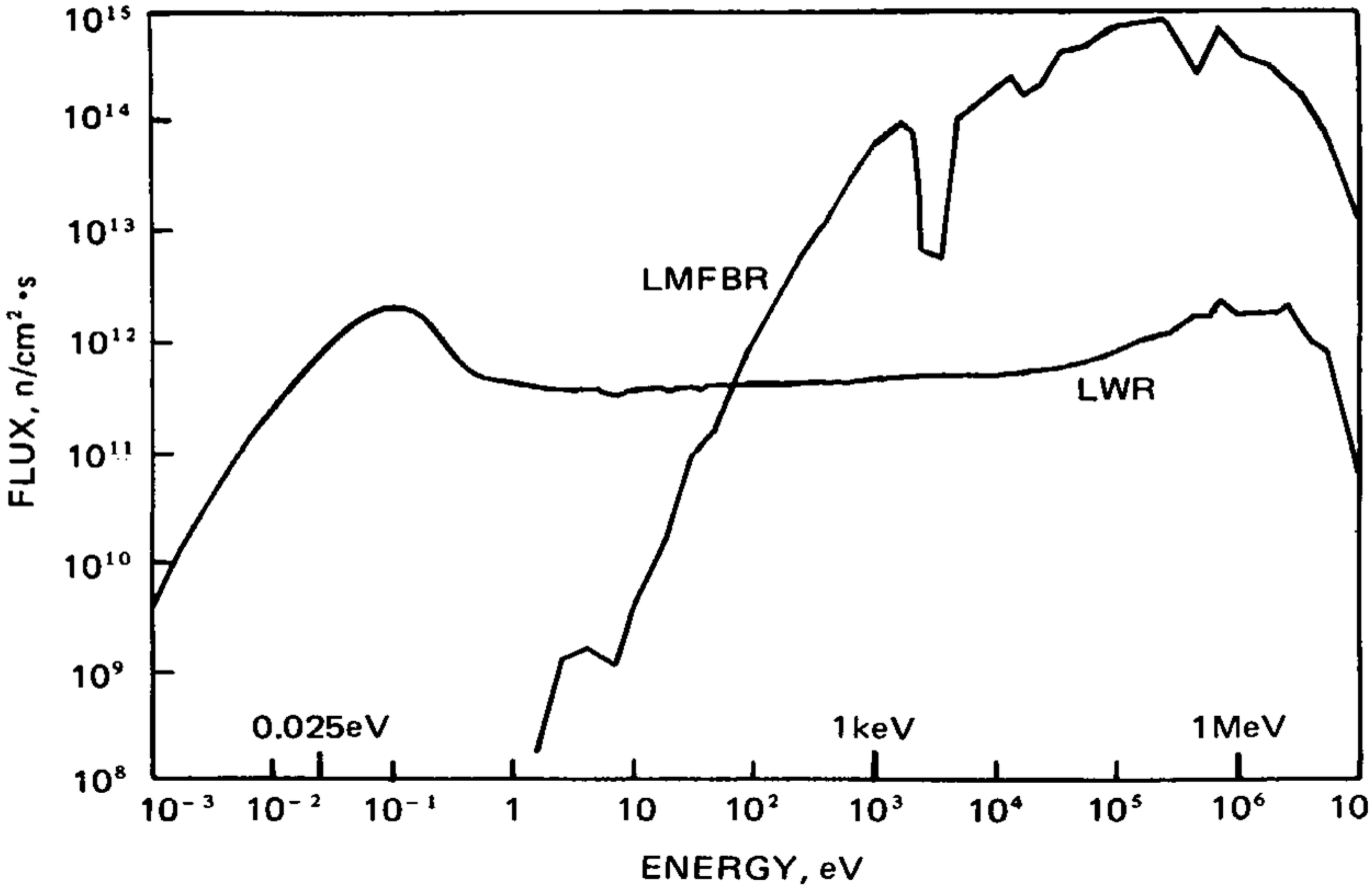
- Production Rate ↔ Absorption Rate
  - Neutron flux and material cross sections are highly dependent on neutron energies
    - Neutron energy affects probability of being absorbed (cross section)
    - Probability of absorption affects density of neutrons with that particular energy (flux)
  - Creates a complex problem to solve
    - Highly-dependent on how neutrons lose energy (slow down) through scattering in a material.

Cross Section (Energy Dependence)





Neutron Fluxes

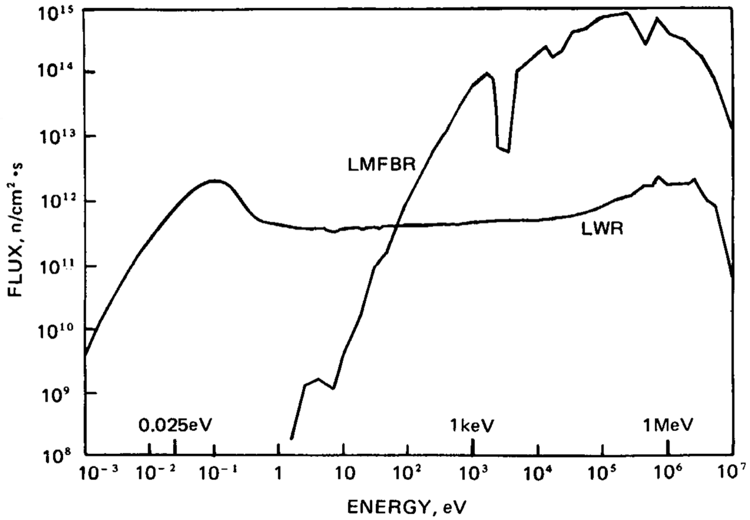


One-Energy Group Averaged Model

ONE-SPEED / -GROUP

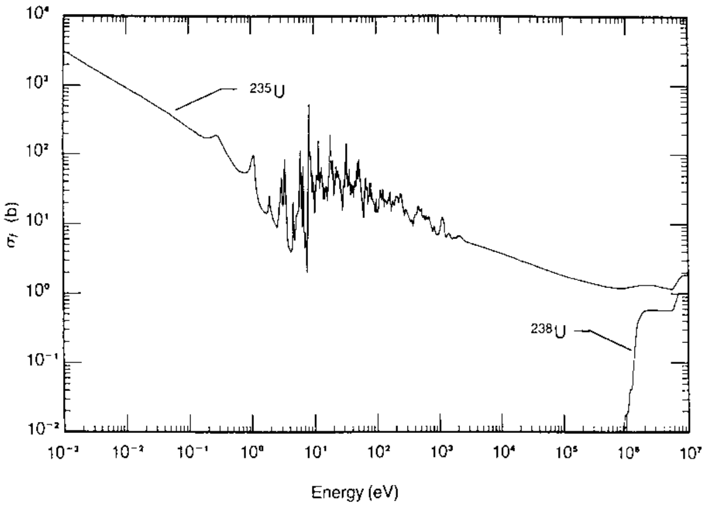
Flux

$$\Phi = \int_0^\infty dE\, \Phi(E)$$



Cross Sections

$$\Sigma_r = \frac{\int_0^\infty dE\, \Sigma_r(E)\Phi(E)}{\int_0^\infty dE\, \Phi(E)}$$



Infinite System

- Neutron Balance
  - Production Rate ↔ Absorption Rate

$$\nu\Sigma_f\Phi \qquad\qquad \Sigma_a\Phi$$

  - (Infinite) Multiplication Factor

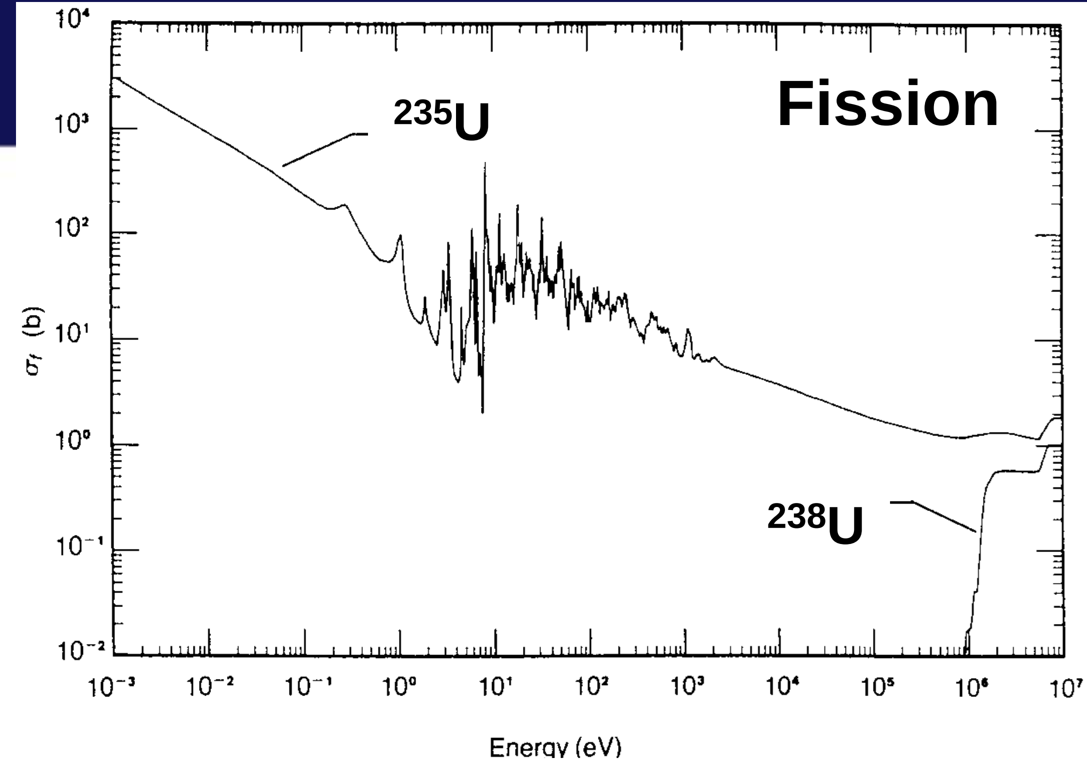
$$\boxed{k_\infty = \frac{\text{Production Rate}}{\text{Absorption Rate}} = \frac{\nu\Sigma_f}{\Sigma_a}}$$

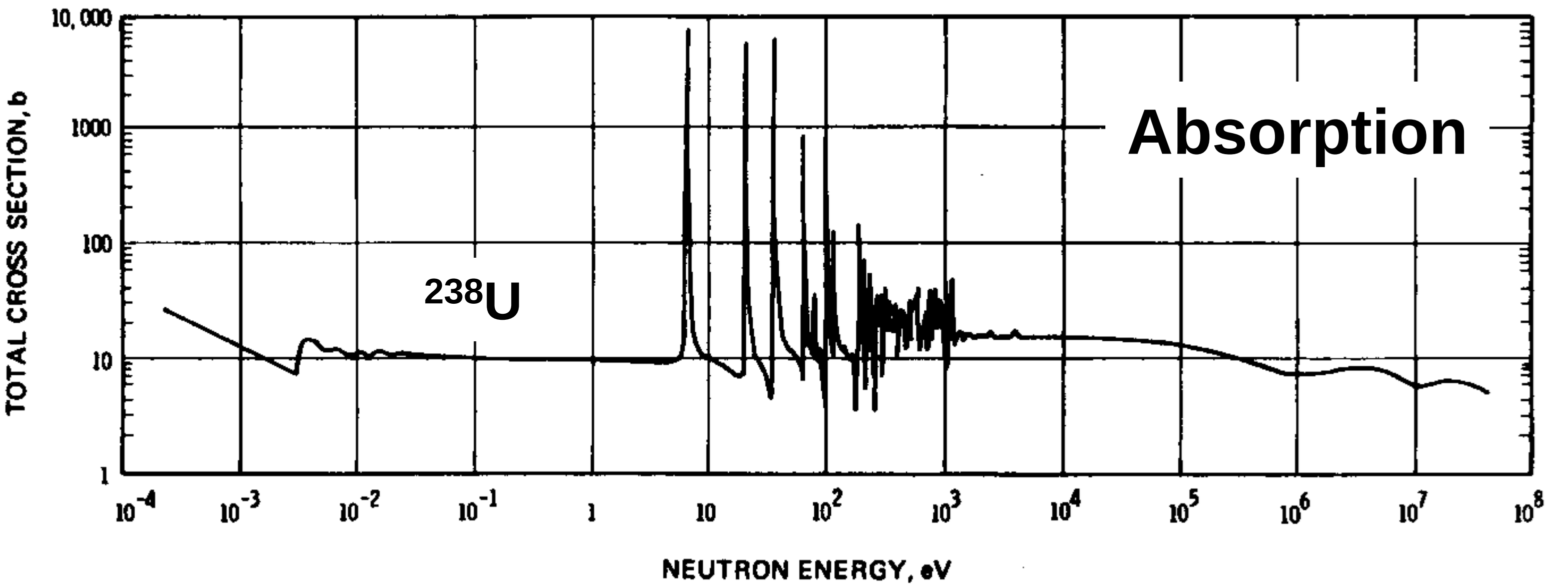
  - Simplified model using one energy group
  - Everything has been effectively energy averaged

Infinite System

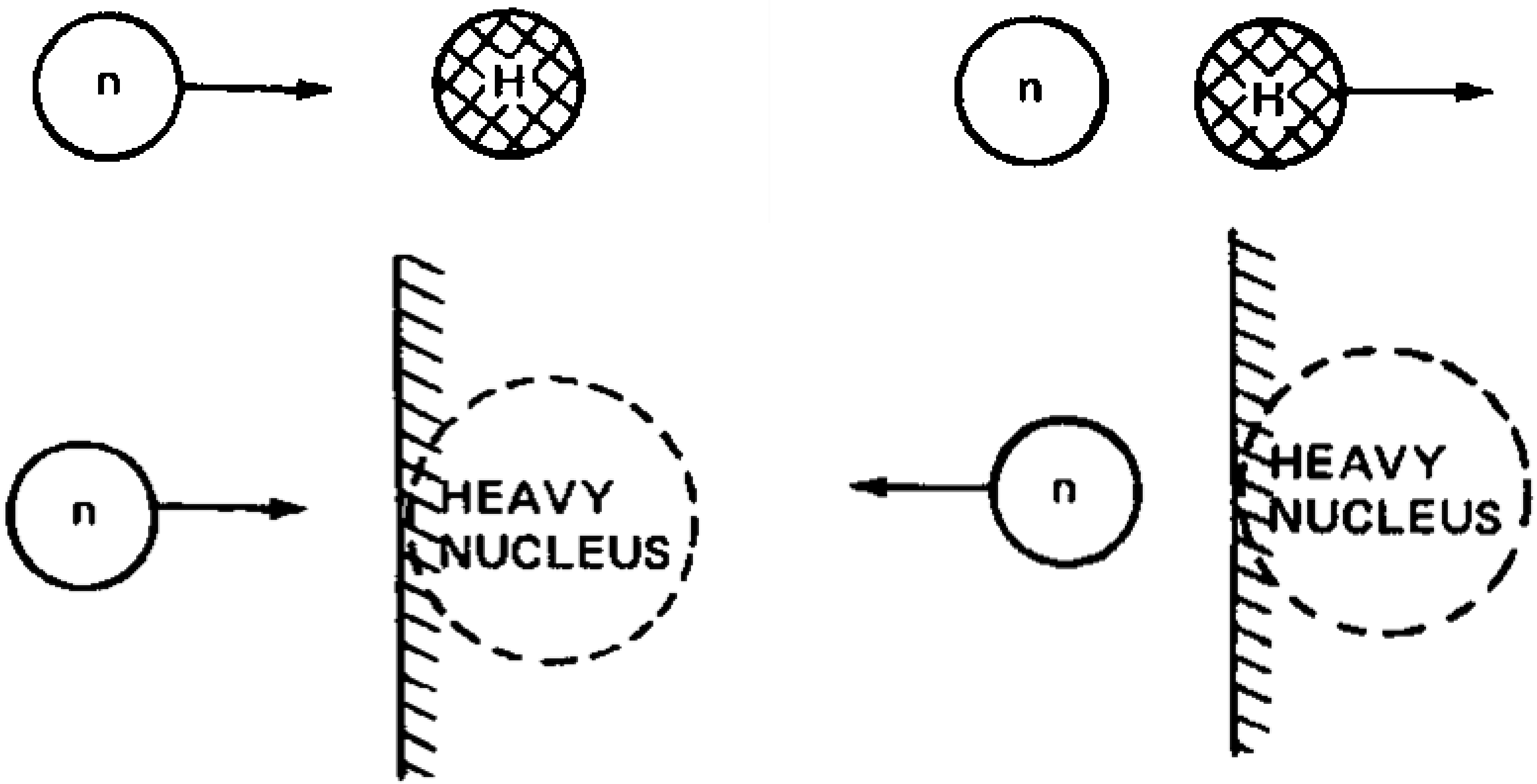
- For fissile isotopes (U235, Pu239, etc) fission is most efficiently caused by thermal neutrons (energy < 1 eV)
- However, neutrons produced by fission are born with high energy (energy > 2 MeV)
- In order for the chain reaction to continue these high-energy fission neutrons must be slowed down to thermal energies (7 orders of magnitude)
  - Neutrons can lose energy through elastic collisions with target atoms in the material.
  - We also want to minimize the number of neutrons that are absorbed before they reach thermal energies and can cause fission events.

Cross Section (Energy Dependence)





Neutron Scattering



Explain the principle of neutron moderation by light nuclei and the importance to thermal reactors

Neutron Moderation

- The process of slowing fast fission neutrons down to thermal energies is called **moderation**.
- Elastic Scattering Kinetics

$$\Delta E_{\max} = E(1-\alpha)$$

$$\alpha = \left(\frac{A-1}{A+1}\right)^2$$

Maximum Change in Neutron Energy per Collision

$$E_{\min} = E - \Delta E_{\max} = \alpha E$$

$$\frac{E_{\min}}{E} = \alpha$$

Minimum Neutron Energy Following a Collision

Neutron Moderator Materials

Neutron Moderation Properties of Selected Atoms and Molecules

Moderator Target	Atomic Mass (A)	Scattering Ratio (α)	(1- α)	Collisions to Thermal[†]
H	1	0.000	1.000	18
H2O				20
D	2	0.111	0.889	25
D2O				35
Be	9	0.640	0.360	86
C	12	0.716	0.284	114
O	16	0.779	0.221	150
Na	23	0.840	0.160	218
U	238	0.983	0.017	2148

[†] Average number of collisions to moderate a fast (1 MeV) neutron to a thermal equilibrium energy of 0.025 eV.

- Low Z atoms are more effective moderators than high Z atoms
- Most modern reactors use H, D, or C as moderators

Moderator Materials

- Hydrogen
  - Highest average energy loss per collision of any target atom.
  - No "backscattering," several collisions are required to reflect a neutron's direction 180° .
  - Small, but noticeable, absorption cross section
- Deuterium
  - Almost as effective as hydrogen-1 at moderation.
  - Smaller absorption cross section than H11.

Moderator Materials

- Increasingly High-Z Materials
  - Lower Average Energy Loss $\rightarrow$ Decreasingly Effective Moderation
  - Backscatter $\rightarrow$ Increasingly Effective Reflection

Moderator Materials

- Water
  - Efficient / Small Reactor Core
  - Absorption
- Deuterium / Beryllium / Graphite
  - Increasingly Larger Cores
- Sodium
  - Moderation / Absorption / T-H Trade-Offs
- Heavy Metals
  - Fast Reactor Designs

Infinite Systems

- Consider the life cycle of a single fission neutron, and the different paths it can take:
- Born at high energy (fast > 1MeV)
  - Some fast neutrons are absorbed and cause fission
- Interacts with moderator to slow down
  - Some are absorbed by moderator
- Once the neutron reaches thermal energy it is absorbed
  - Only some of the thermal neutrons are absorbed in the fuel.
  - Only some of the thermal neutrons absorbed in the fuel cause fission events

Identify the terms in the four and six factor formulas

Infinite Systems

- **Four-Factor Formula** for k-infinity

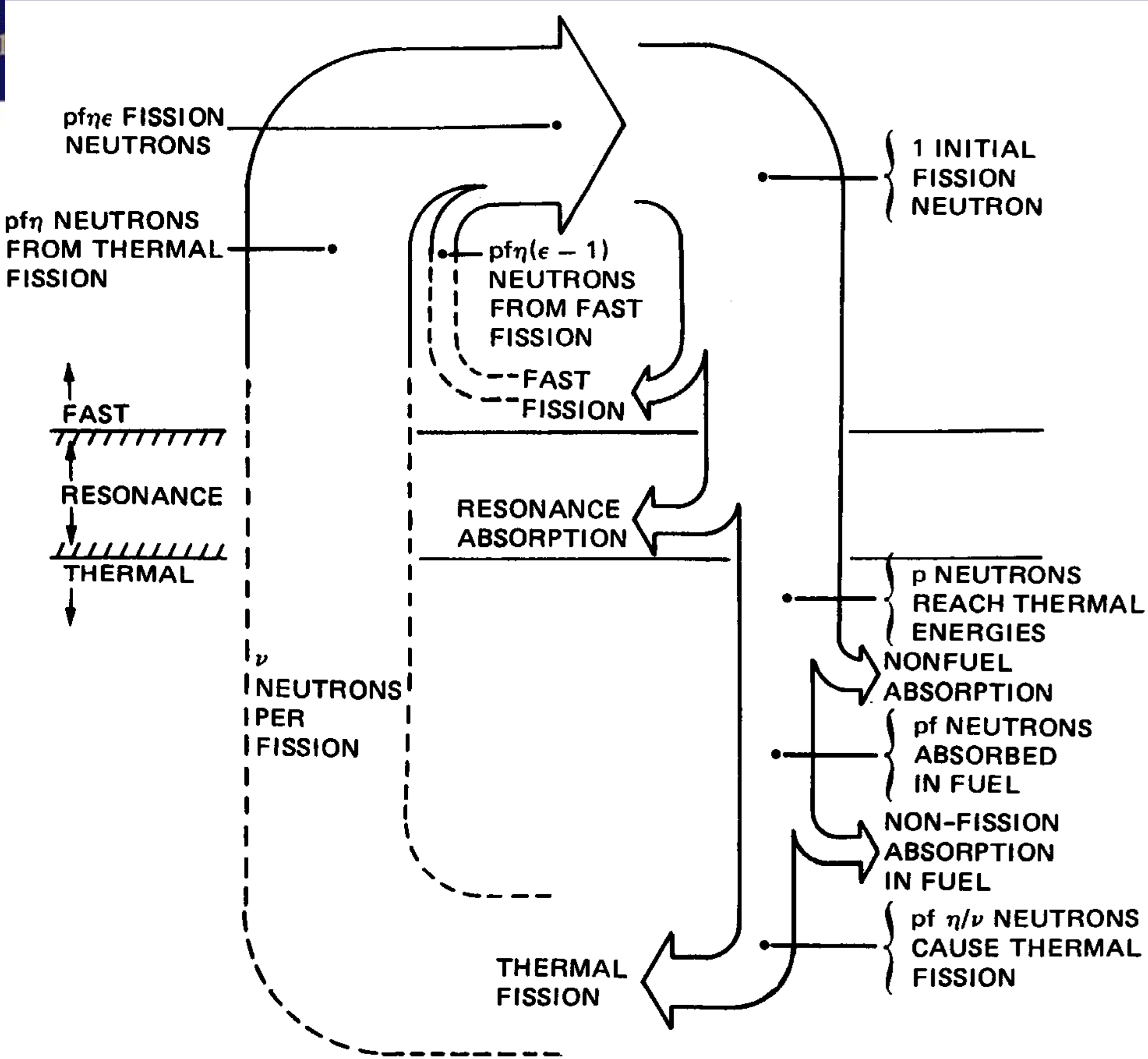
$$k_\infty = \varepsilon p \, \eta f$$

$$\varepsilon = \text{Fast Fission Factor} = \frac{\left(\nu\Sigma_{\mathrm{f}}\right)_{total}}{\left(\nu\Sigma_{\mathrm{f}}\right)_{th}}$$

$$p = \text{Resonance Escape Probability} = \frac{\left(\Sigma_{\mathrm{a}}\right)_{th}}{\left(\Sigma_{\mathrm{a}}\right)_{total}}$$

$$f = \text{Thermal Utilization Factor} = \frac{\left(\Sigma_{\mathrm{a}}\right)_{th}^{fuel}}{\left(\Sigma_{\mathrm{a}}\right)_{th}}$$

$$\eta = "Eta" = \text{Reproduction Factor} = \frac{\left(\nu\Sigma_{\mathrm{f}}\right)_{th}}{\left(\Sigma_{\mathrm{a}}\right)_{th}^{fuel}}$$


pfηε FISSION NEUTRONS
pfη NEUTRONS FROM THERMAL FISSION
FAST
RESONANCE
THERMAL
pfη(ε − 1) NEUTRONS FROM FAST FISSION
FAST FISSION
RESONANCE ABSORPTION
ν NEUTRONS PER FISSION
THERMAL FISSION
1 INITIAL FISSION NEUTRON
p NEUTRONS REACH THERMAL ENERGIES
NONFUEL ABSORPTION
pf NEUTRONS ABSORBED IN FUEL
NON-FISSION ABSORPTION IN FUEL
pf η/ν NEUTRONS CAUSE THERMAL FISSION

Infinite Systems

- Four-Factor Formula

$$k_\infty = \varepsilon\, p\, \eta\, f$$

$$\boxed{k_\infty = \frac{(\nu\Sigma_f)_{total}}{(\Sigma_a)_{total}}}$$

$$\varepsilon = \text{Fast Fission Factor} = \frac{(\nu\Sigma_f)_{total}}{\cancel{(\nu\Sigma_f)_{th}}}$$

$$p = \text{Resonance Escape Probability} = \frac{\cancel{(\Sigma_a)_{th}}}{(\Sigma_a)_{total}}$$

$$f = \text{Thermal Utilization Factor} = \frac{\cancel{(\Sigma_a)_{th}^{fuel}}}{\cancel{(\Sigma_a)_{th}}}$$

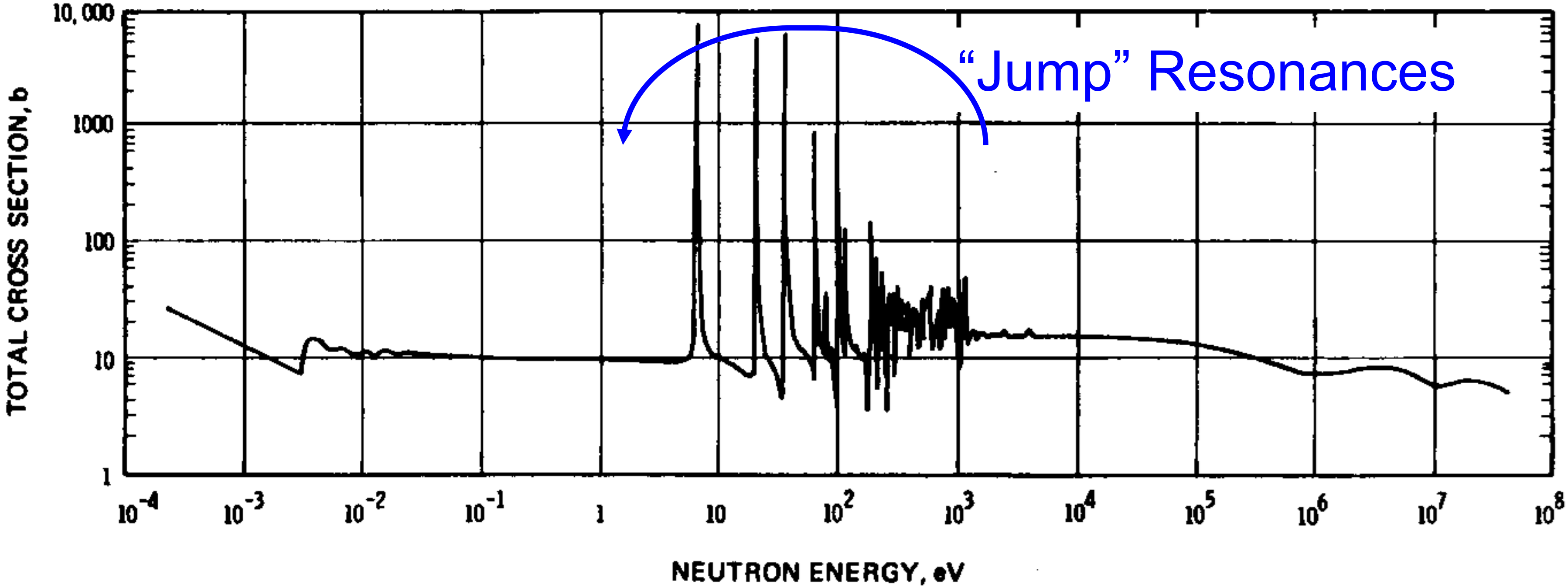
$$\eta = \text{"}Eta\text{"} = \text{Reproduction Factor} = \frac{\cancel{(\nu\Sigma_f)_{th}}}{\cancel{(\Sigma_a)_{th}^{fuel}}}$$

$$\left.\begin{matrix} f \\ \eta \end{matrix}\right\rangle\ \eta\, f = \frac{(\nu\Sigma_f)_{th}}{(\Sigma_a)_{th}}$$

Four Factor Formula

- Probabilistic Model
  - Considers Thermal-Neutron Fission From Fast (Fission Spectrum) Neutrons
    - e Fast Fission Fractional Augmentation (e > 1)
    - p Fraction of Neutrons Reaching Thermal (Not Absorbed Fast or [Primarily] in Resonances)
    - f Fraction Absorbed in Fuel (U + Pu)
    - h Neutrons Produced per Thermal Neutron Absorbed in Fuel

^{238}U Absorption Cross Section
Importance of Resonance Escape



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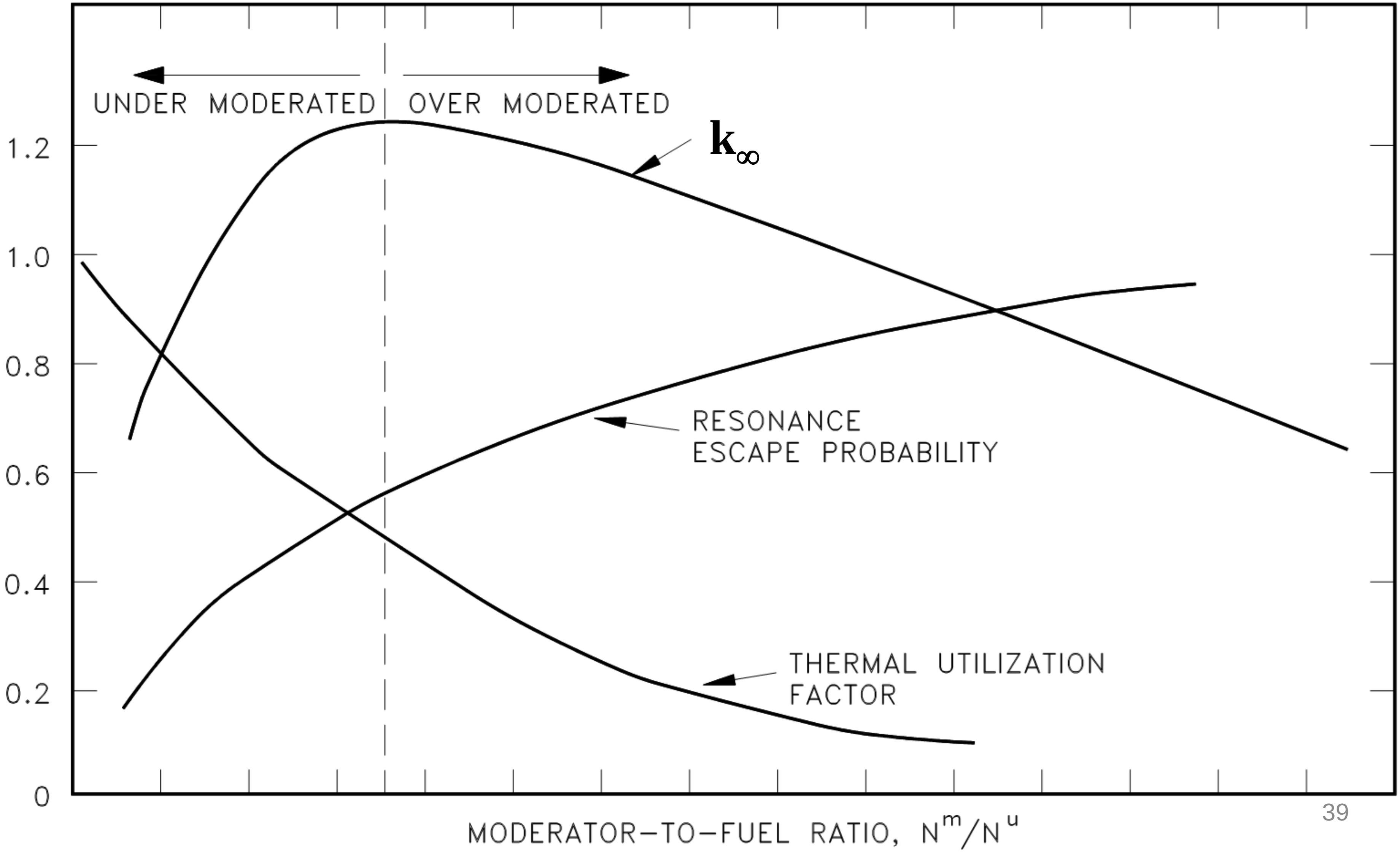
Moderator-to-Fuel Ratio Effect on k_∞



Understand the impact of heterogeneity on neutron balance

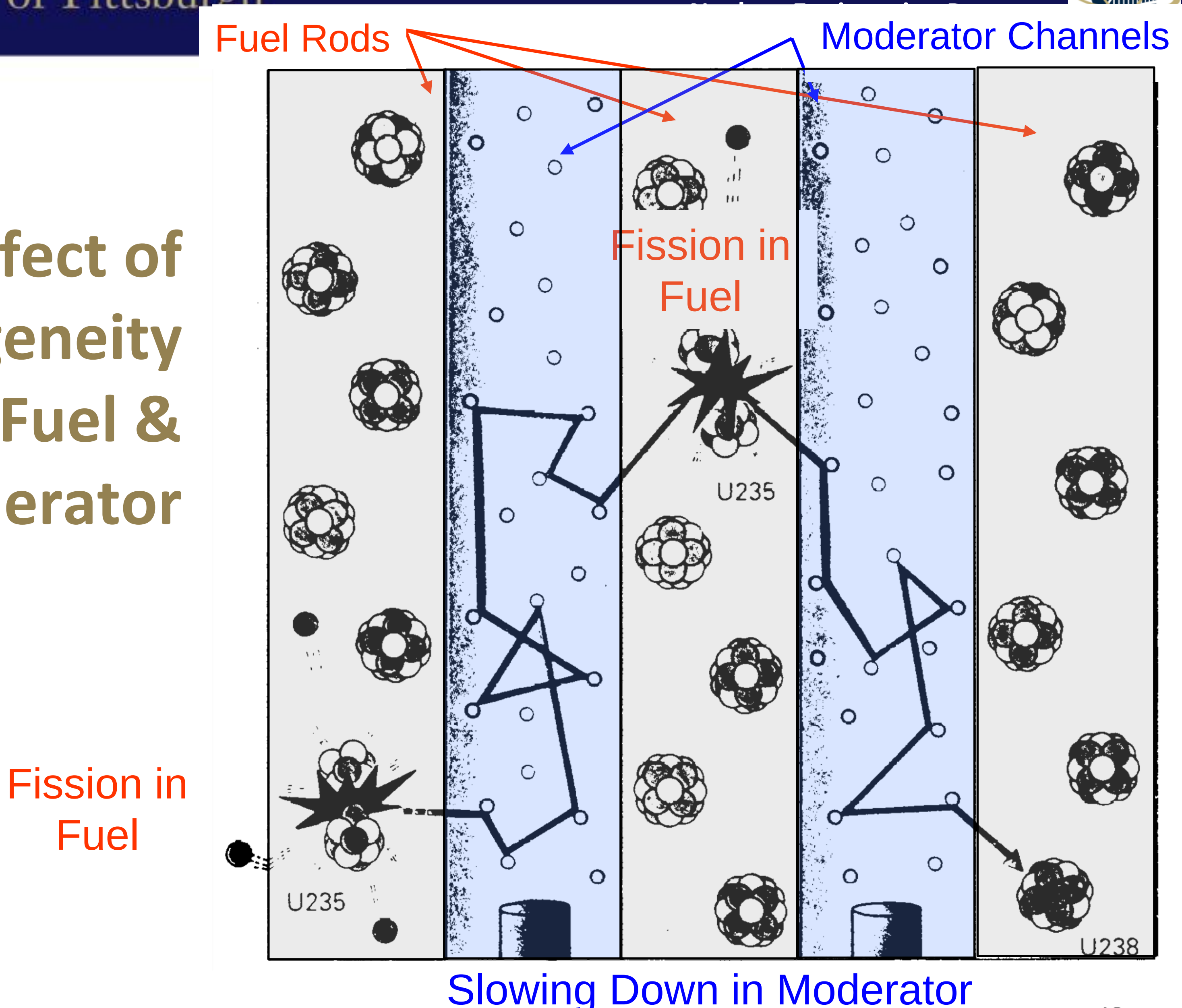
Heterogeneous Systems

- Heterogeneous Systems
  - Lumping Fuel
    - Increases Resonance Escape Probability p
    - Decrease Thermal Utilization Factor f
  - Optimization

    $$\Delta p > \Delta f$$

  - Natural Uranium / Graphite Critical
  - LWR Fuel Pin Lattices

Effect of Heterogeneity of Fuel & Moderator



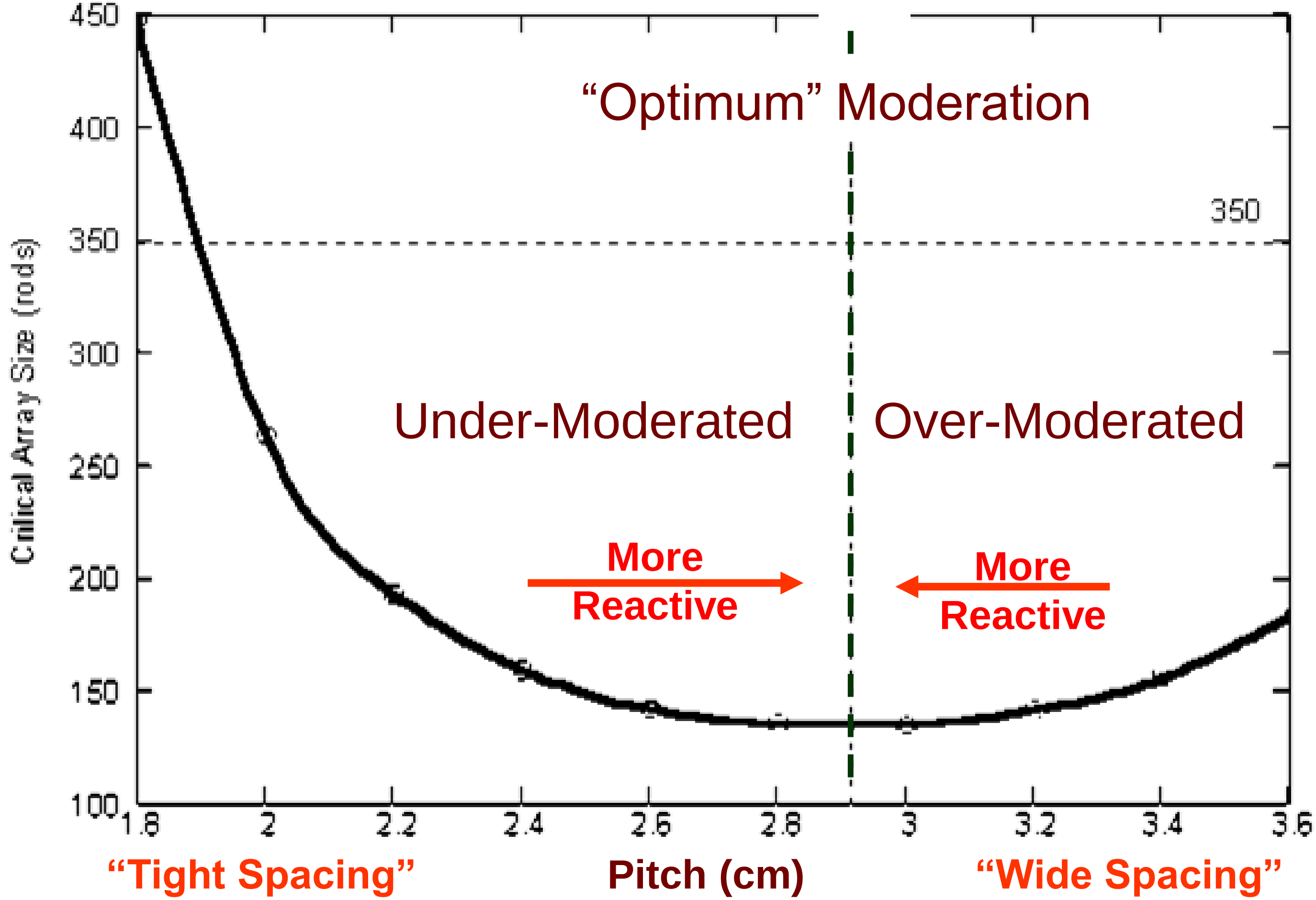
Four-Factor Formula

- Fuel "Lumping" / Lattice Arrangement
  - Increase Fast Fission Factor e
  - Increase Resonance Escape Probability p
  - Decrease Thermal Utilization f
  - Pin Diameter and Spacing to optimize $p \times f$
    - Similar to previous k_∞ vs. M-to-F curve
  - Example: LWR-like lattice

Moderation Effects

LWR-Like Fuel Pin Lattice (4.3 wt% ^{235}U)

X-Axis Nominal (Increasing Moderator-to-Fuel Ratio)



Differentiate between the infinite (k_∞) and effective (keff or k) multiplication factors

Finite System

- Neutron Balance

Production ↔ Absorption + Leakage

$$\nu\Sigma_f \Phi \leftrightarrow \Sigma_a \Phi \quad + \text{Leakage}$$

$$k_{eff} = k = \frac{\text{Production}}{\text{Absorption} + \text{Leakage}}$$

Note: $k_\infty > k_{\text{eff}}$ (To accommodate leakage)

Six Factor Formula

- Six-Factor Formula

$$k_{\text{eff}} = \mathrm{k}_{\infty} P_{\text{fnl}} P_{\text{tnl}}$$

$$k_{\text{eff}} = \varepsilon\, p\, \eta\, f\, P_{\text{fnl}} P_{\text{tnl}}$$

P_{fnl} = Fast Non-Leakage Probability

P_{tnl} = Thermal Non-Leakage Probability

$$k_{\text{eff}} = \varepsilon\, p\, \eta\, f\, P_{\text{nl}}$$

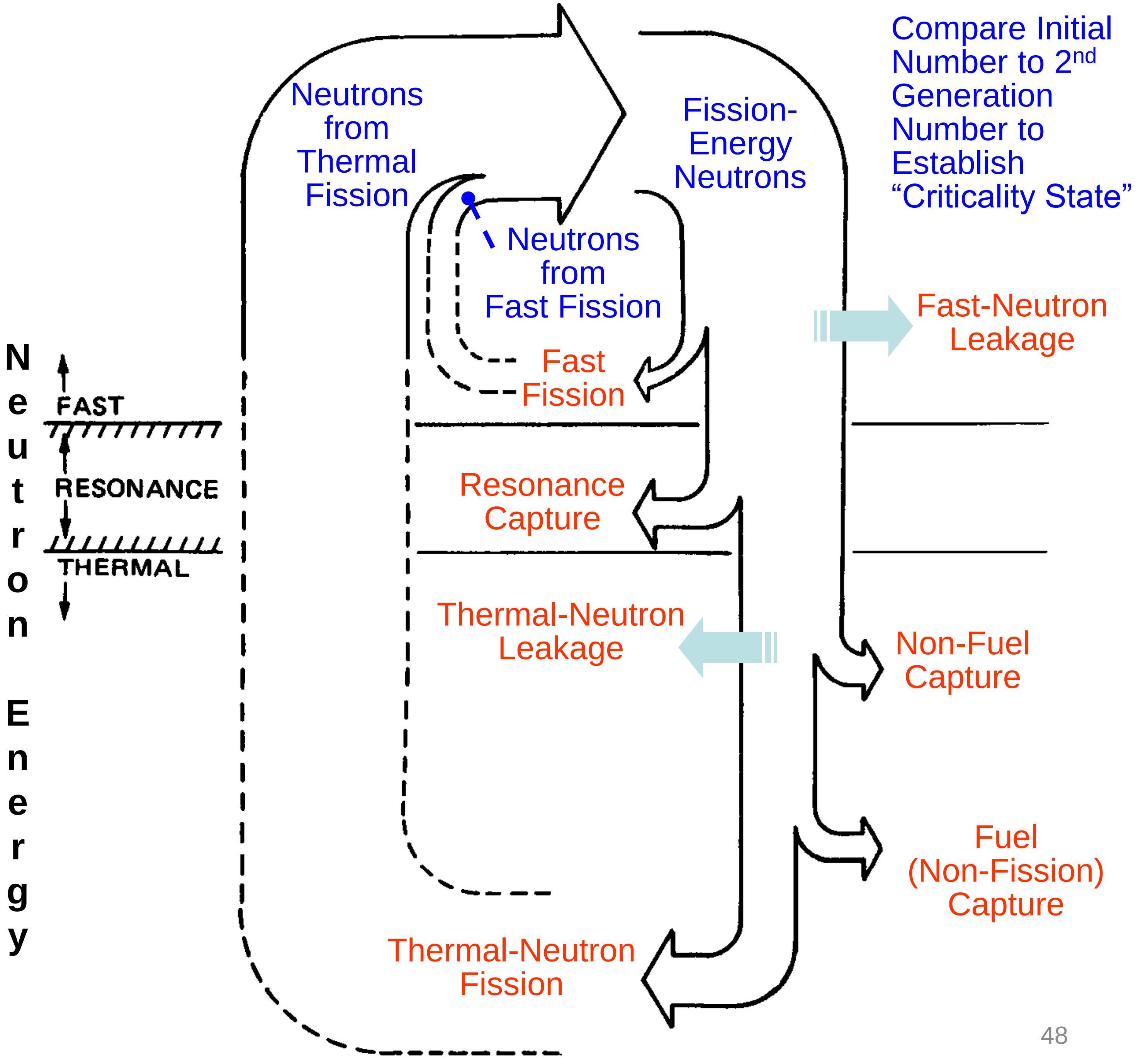
P_{nl} = Total Non-Leakage Probability

Conceptual Neutron Flow

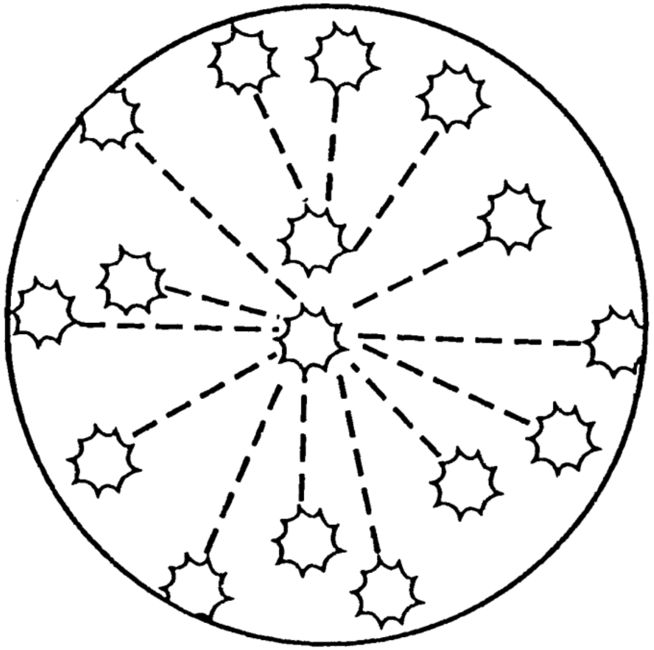
"4- Factor" Model

+ Leakage

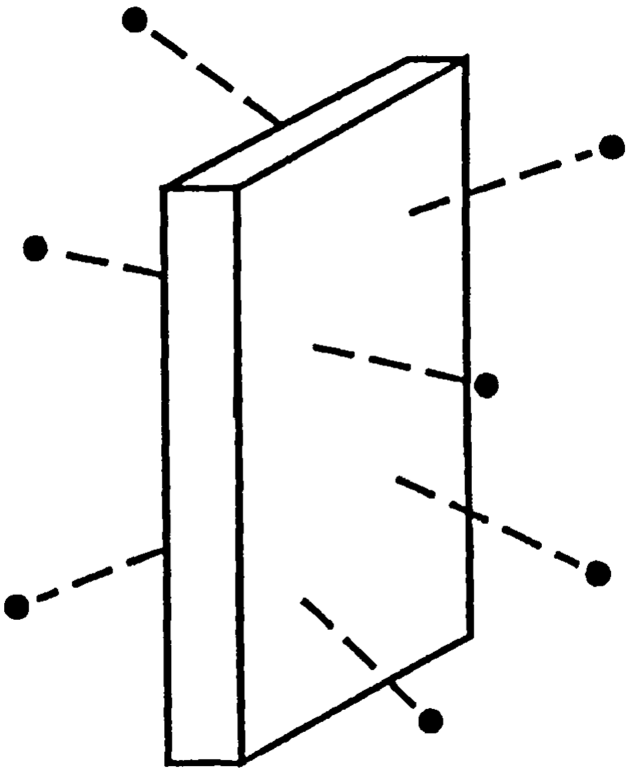
"6- Factor" Model



Neutron Leakage Effect



Few Neutrons Leak from Volume as Sphere



More Neutrons Leak from Volume as Slab

Leakage depends on shape & size (surface-to-volume ratio)