



University of Pittsburgh

ME/ENGR 2100

Fundamentals of Nuclear Engineering

Neutron Diffusion:
Non-Multiplying Media

Dr. Daniel F. Gill

Stephen R. Tritch Engineering
Swanson School of Engineering
University of Pittsburgh





Related Reading

- Chapter 5 of Duderstadt and Hamilton

OR

- Sections 5.1-5.7 Lamarsh and Baratta
- Sections 6.1-6.4 Lamarsh and Baratta



Learning Objectives

- Calculate the neutron distribution due to an external source in a non-multiplying medium for common geometrical configurations

Diffusion Length

$$-D \frac{d^2}{dx^2} \phi(x) - \Sigma_a \phi(x) = s(x)$$

- Rewrite this equation as

$$\frac{d^2}{dx^2} \phi(x) - \frac{1}{L^2} \phi(x) = -\frac{s(x)}{D}$$

- The variable L is called the diffusion length and L^2 is called the diffusion area

$$L = \sqrt{\frac{D}{\Sigma_a}}$$

$$L^2 = \frac{D}{\Sigma_a}$$

- The diffusion length has units of cm and the diffusion area of cm^2

Diffusion Length

- It can be shown that

$$L^2 = \frac{1}{6} \langle r^2 \rangle$$

- L is $1/\sqrt{6}$ of the root mean square distance to absorption
- L measures the distance to which the neutron will diffuse (on average) away from the source before being absorbed
 - Not the same as the total path length since the neutron will undergo many reactions between being emitted from the source and being absorbed
 - Also much larger than a mean free path

Differential Equation Solution

- The differential equation

$$\frac{d^2}{d^2x} \phi(x) - \frac{1}{L^2} \phi(x) = -\frac{s(x)}{D}$$

has the following solution

$$\phi(x) = \phi_{\text{homogeneous}}(x) + \phi_{\text{particular}}(x)$$

$$\phi_{\text{homogeneous}}(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

where the homogeneous solution must satisfy the boundary conditions and the particular solution our source term

Example 1: Semi-Infinite Slab

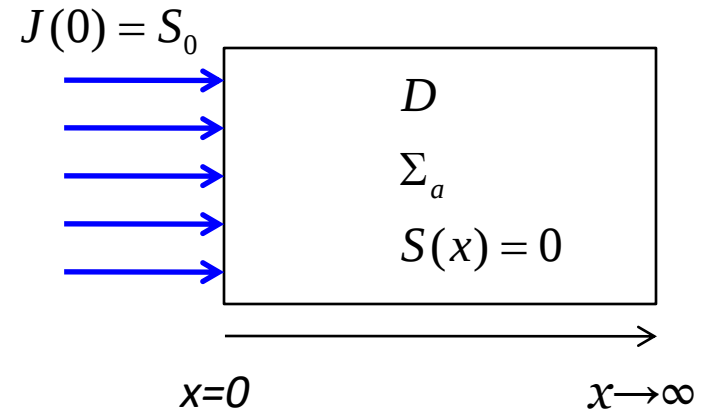
- No particular solution in this case

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

- Boundary Conditions

$$J(0) = S_0$$

$$\lim_{x \rightarrow \infty} \phi(x) < \infty$$



Example 1: Semi-Infinite Slab

- Finite Flux

$$\lim_{x \rightarrow \infty} (c_1 e^{-x/L} + c_2 e^{x/L}) < \infty$$

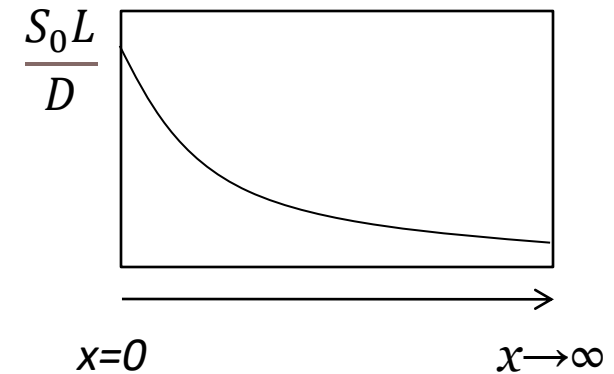
$$c_2 = 0$$

- Incident Current

$$J(0) = -D \left. \frac{d\phi(x)}{dx} \right|_{x=0} = S_0$$

$$-D \left. \frac{d\phi(x)}{dx} \right|_{x=0} = D \left. \frac{c_1 e^{-x/L}}{L} \right|_{x=0} = S_0$$

$$c_1 = \frac{S_0 L}{D}$$



- Flux Solution

$$\phi(x) = \frac{S_0 L}{D} e^{-x/L}$$

Example 2: Finite Slab

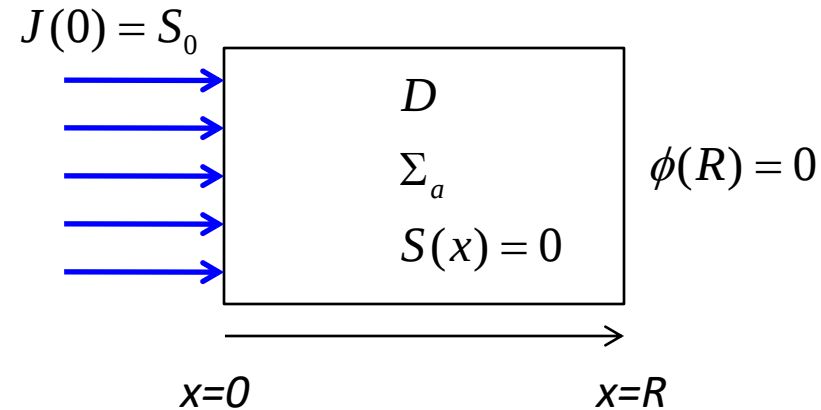
- Again, no particular solution

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

- Boundary Conditions

$$J(0) = S_0$$

$$\phi(R) = 0$$



Example 2: Finite Slab

$$J(0) = -D \left. \frac{d\phi(x)}{dx} \right|_{x=0} = S_0$$

$$-D \left. \frac{d}{dx} (c_1 e^{-x/L} + c_2 e^{x/L}) \right|_{x=0} = S_0$$

$$-D \left(\frac{-c_1}{L} e^{-x/L} + \frac{c_2}{L} e^{x/L} \right) \bigg|_{x=0} = S_0$$

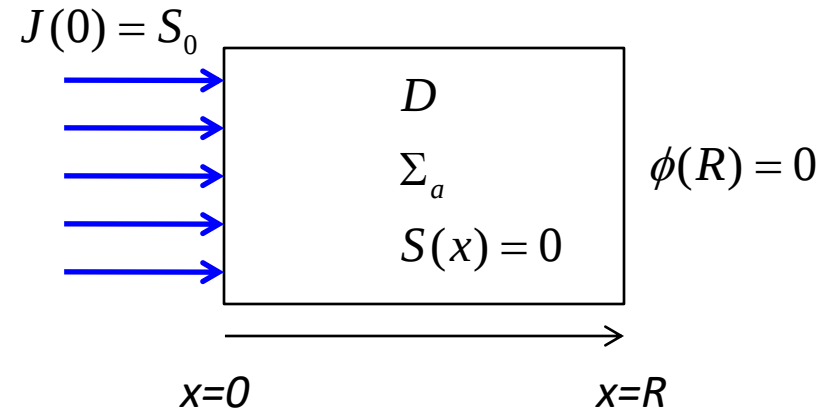
$$(c_1 - c_2) = \frac{S_0 L}{D}$$

$$c_1 = c_2 + \frac{S_0 L}{D}$$

$$\phi(x) = \left(c_2 + \frac{S_0 L}{D} \right) e^{-x/L} + c_2 e^{x/L}$$

$$\phi(x) = \frac{S_0 L}{D} e^{-x/L} + c_2 (e^{-x/L} + e^{x/L})$$

$$\phi(x) = \frac{S_0 L}{D} e^{-x/L} + 2c_2 \cosh\left(\frac{x}{L}\right)$$



Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Example 2: Finite Slab

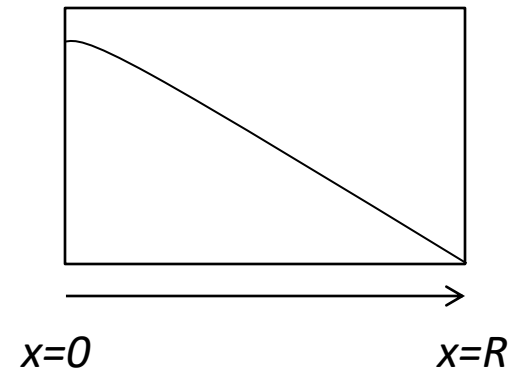
$$\phi(R) = 0$$

$$\phi(R) = \frac{S_0 L}{D} e^{-R/L} + 2c_2 \cosh(R/L) = 0$$

$$c_2 = \frac{-S_0 L e^{-R/L}}{2D \cosh(R/L)}$$

$$\phi(x) = \frac{S_0 L}{D} e^{-x/L} + 2 \left(\frac{-S_0 L e^{-R/L}}{2D \cosh(R/L)} \right) \cosh\left(\frac{x}{L}\right)$$

$$\phi(x) = \frac{S_0 L}{D} \left(e^{-x/L} - e^{-R/L} \frac{\cosh(x/L)}{\cosh(R/L)} \right) = \frac{S_0 L}{D} \left(\frac{\sinh((R-x)/L)}{\cosh(R/L)} \right)$$



Example 3: Finite Slab, with source

- Need to find particular solution

$$\frac{d^2}{dx^2} \phi(x) - \frac{1}{L^2} \phi(x) = -\frac{s(x)}{D}$$

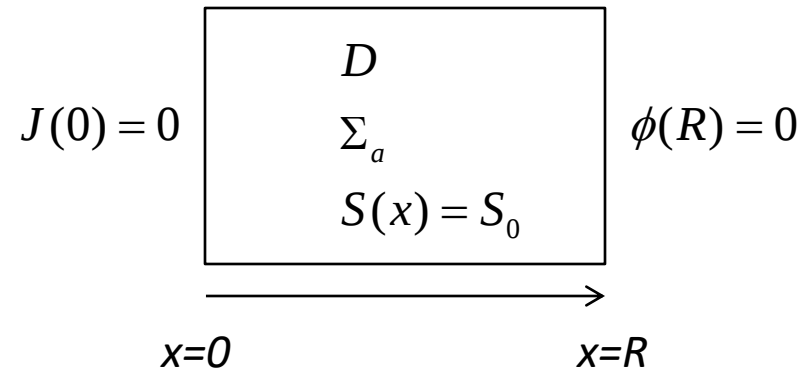
$$\phi(x) = \phi_{\text{homogeneous}}(x) + \phi_{\text{particular}}(x)$$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L} + \phi_{\text{particular}}(x)$$

- Boundary Conditions

$$J(0) = 0$$

$$\phi(R) = 0$$



We will first find particular solution, guessing a constant because our source is constant

Example 3: Finite Slab, with source

$$\phi_{\text{particular}}(x) = F$$

$$\frac{d^2}{dx^2} \phi_{\text{particular}}(x) - \frac{1}{L^2} \phi_{\text{particular}}(x) = -\frac{S_0}{D}$$

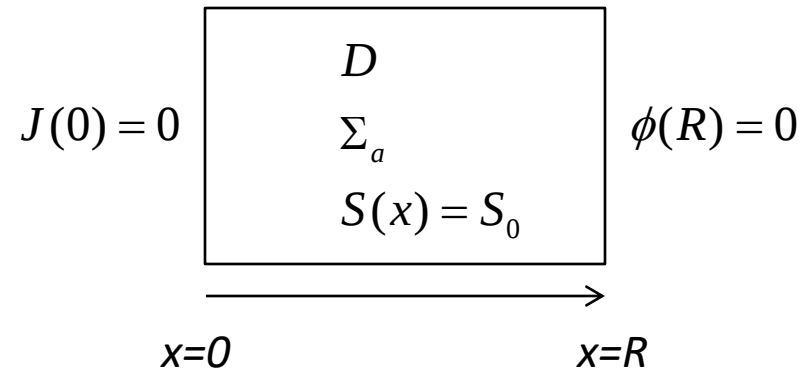
$$\frac{d^2}{dx^2} F - \frac{1}{L^2} F = -\frac{S_0}{D}$$

$$-\frac{1}{L^2} F = -\frac{S_0}{D}$$

$$F = \frac{S_0 L^2}{D} = \frac{S_0}{\Sigma_a}$$

$$\phi_{\text{particular}}(x) = \frac{S_0}{\Sigma_a}$$

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L} + \frac{S_0}{\Sigma_a}$$



We will first find particular solution, guessing a constant because our source is constant

Example 3: Finite Slab, with source

$$J(0) = -D \left. \frac{d\phi(x)}{dx} \right|_{x=0} = 0$$

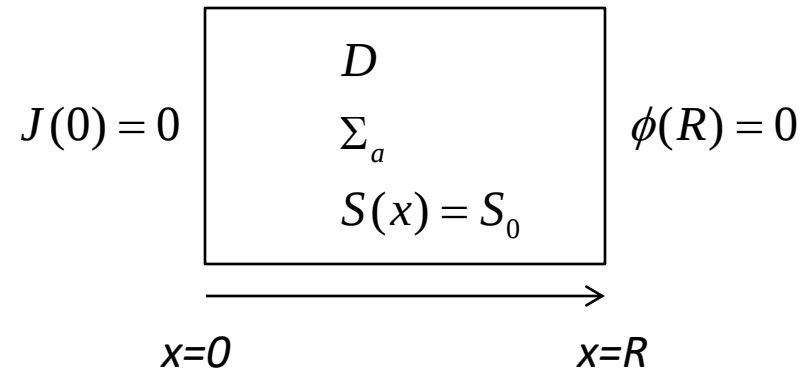
$$J(0) = -D \left. \frac{d}{dx} \left(c_1 e^{-x/L} + c_2 e^{x/L} + \frac{S_0}{\Sigma_a} \right) \right|_{x=0} = 0$$

$$-D \left(\frac{-c_1}{L} e^{-x/L} + \frac{c_2}{L} e^{x/L} \right) \bigg|_{x=0} = 0$$

$$c_1 = c_2$$

$$\phi(x) = c_1 e^{-x/L} + c_1 e^{x/L} + \frac{S_0}{\Sigma_a}$$

$$\phi(x) = 2c_1 \cosh(x/L) + \frac{S_0}{\Sigma_a}$$



Example 3: Finite Slab, with source

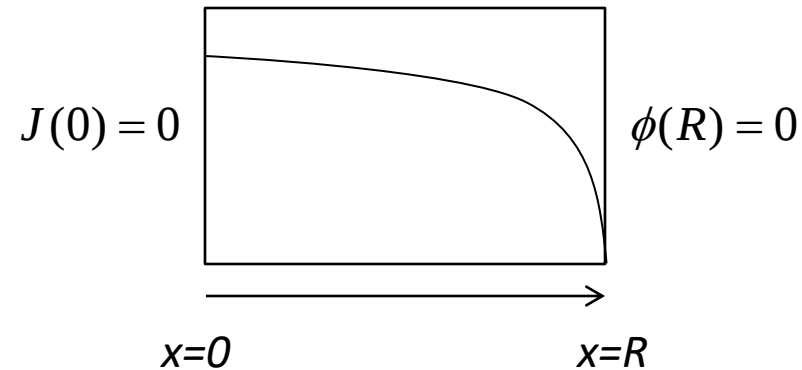
$$\phi(R) = 0$$

$$\phi(R) = 2c_1 \cosh(R / L) + \frac{S_0}{\Sigma_a} = 0$$

$$c_1 = \frac{-S_0}{2\Sigma_a \cosh(R / L)}$$

$$\phi(x) = 2 \left(\frac{-S_0}{2\Sigma_a \cosh(R / L)} \right) \cosh(x / L) + \frac{S_0}{\Sigma_a}$$

$$\phi(x) = \frac{S_0}{\Sigma_a} \left(1 - \frac{\cosh(x / L)}{\cosh(R / L)} \right)$$



Example 4: Point Source

- Consider a point source emitting Q neutrons / sec (isotropically) in an infinite medium.
 - Only dependence on r (no angular dependence)
 - In spherical coordinates

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi(r)}{dr} - \frac{1}{L^2} \phi(r) = 0, \quad r > 0$$

- Boundary Conditions:

$$\lim_{r \rightarrow 0} 4\pi r^2 J(r) = Q$$

$$\lim_{r \rightarrow \infty} \phi(r) < \infty$$

Example 4: Point Source

- General solution to this problem is given by

$$\phi(r) = c_1 \frac{e^{-r/L}}{r} + c_2 \frac{e^{r/L}}{r}$$

- Applying finite flux boundary condition

$$\phi(r) = c_1 \frac{e^{-r/L}}{r}$$

Example 4: Point Source

- Applying current boundary condition

$$\begin{aligned} J(r) &= -D \frac{d\phi(r)}{dr} = -Dc_1 \frac{d}{dr} \frac{e^{-r/L}}{r} \\ &= -Dc_1 \left(\frac{-\frac{r}{L} e^{-r/L} - e^{-r/L}}{r^2} \right) = Dc_1 \left(\frac{1}{rL} + \frac{1}{r^2} \right) e^{-r/L} \end{aligned}$$

$$\lim_{r \rightarrow 0} 4\pi r^2 J(r) = \lim_{r \rightarrow 0} 4\pi Dc_1 \left(\frac{r}{L} + 1 \right) e^{-r/L} = Q$$

Example 4: Point Source

- Resulting expression for C_1

$$C_1 = \frac{Q}{4\pi D}$$

- Making the flux due to a point source in an infinite medium equal to

$$\phi(r) = \frac{Qe^{-r/L}}{4\pi Dr}$$

- Different from the inverse square law we are accustomed to for point sources. Why? Does this describe the flux in a vacuum due to a point source?