University of Pittsburgh

ME/ENGR 2100 Fundamentals of Nuclear Engineering

Neutron Diffusion

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Related Reading

- Chapter 5 of Duderstadt and Hamilton

OR

- Sections 5.1-5.7 Lamarsh and Baratta
- Sections 6.1-6.4 Lamarsh and Baratta

Learning Objectives

- Define the scalar neutron flux and net neutron current and provide a physical interpretation of each quantity
- Write the mathematical conservation relationship describing the “exact” neutron balance (continuity equation) in non-multiplying systems
- Write the neutron diffusion equation describing neutron balances (gains and losses) in non-multiplying systems
- Explain under what conditions the diffusion approximation to the continuity equation is valid and how its use in reactor analysis is justified
- Calculate the neutron distribution due to an external source in a non-multiplying medium for common geometrical configurations
- Calculate the fundamental mode (scalar flux and multiplication factor) for a given power based on the one-group reactor equation for common geometrical configurations. Be able to find the reactor dimensions which will establish criticality for a given material composition.

Cross Section Review

- Microscopic
  - Probability per unit area that an incident neutron of given energy and direction will interact with a specific nucleus
  - Units of cm^2 or barns (10^{-24} cm^2)
- Macroscopic
  - Probability per unit length that a specific neutron of known energy and direction will interact
  - Units of 1/cm
- Reaction Rate
  - Rate of neutron reactions (per unit volume and time) in a material
  - Units of [Reactions/cm^3/sec]

Neutron Phase Space

- The location of any neutron is described *uniquely* by 7 independent phase variables
  - Spatial Location: $(x, y, z) = \vec{r}$
  - Direction of travel: $\hat{\Omega} = \left(v_x, v_y, v_z\right) / \| v \|$
  - Energy (velocity): E
  - Time: t
- These phase space variables are the independent variables in the equation describing the true balance of neutrons in a system
- To simplify the problem we will assume all neutrons have a single energy and that the same number of neutrons are traveling in each direction

Neutron Density

- Using these assumptions, the neutron density (distribution) at any point in a reactor is

$$n(\vec{r},t)\,dV$$

- This gives the number of neutrons that fall within volume dV at time t.
- Referred to as the *Neutron Density*.
- Sometimes referred to as scalar neutron density to indicate it includes neutrons traveling all directions

Scalar Neutron Flux

- dS is the total path length generated by all neutrons in dV about $(\vec{r})$ during time interval dt.
- Dividing dS by dt gives

$$\phi(\vec{r},t)\,dV = \mathrm{v}n(\vec{r},t)\,dV = \frac{dS}{dt}$$

the rate at which neutrons in dV about point $(\vec{r})$ generate path length at time t.

- $\phi(\vec{r},t)$ is referred to as the scalar neutron flux

Reaction Rate

$$R_t(\vec{r},t)=\Sigma_t(\vec{r},t)\phi(\vec{r},t)$$

- Total Reaction Rate Density
  - Rate at which neutrons at position $\vec{r}$, undergo any type of reaction.

$$R_x(\vec{r},t)=\Sigma_x(\vec{r},t)\phi(\vec{r},t)$$

- Reaction Rate Density
  - Rate at which neutrons at position $\vec{r}$, undergo reaction type *x*.

The equation of continuity

- The equation of continuity is the statement of the fact that since neutrons do not disappear unaccountably, the time rate of change in the number of neutrons in volume V must be accounted for by the known physical processes of production, absorption, and leakage

$$\begin{bmatrix}\text{Rate of change of} \\ \text{number of} \\ \text{neutrons in V}\end{bmatrix} = \begin{bmatrix}\text{Rate of} \\ \text{production of} \\ \text{neutrons in V}\end{bmatrix} - \begin{bmatrix}\text{Rate of} \\ \text{absorption of} \\ \text{neutrons in V}\end{bmatrix} - \begin{bmatrix}\text{Rate of} \\ \text{leakage of} \\ \text{neutrons in V}\end{bmatrix}$$

The equation of continuity

- The *equation of continuity* can be written

$$\int_V \frac{\partial n(\vec{r},t)}{\partial t} dV = \int_V s(\vec{r},t) dV - \int_V \Sigma_a(\vec{r},t)\phi(\vec{r},t) dV - \int_V \left(\nabla \bullet J(\vec{r},t)\right) dV$$

- Because all of the integrals were carried out over the same arbitrary volume the integrands must be equal. Thus

$$\frac{\partial n(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\phi(\vec{r},t) - \nabla \bullet J(\vec{r},t)$$

The equation of continuity

- Using our definition of scalar flux we can rewrite the time rate of change term:

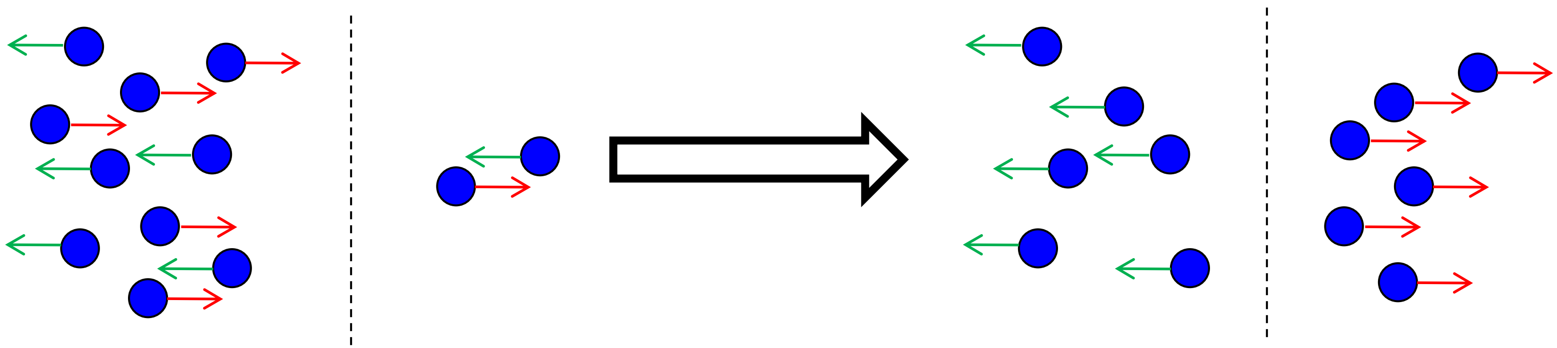
$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\phi(\vec{r},t) - \nabla \bullet J(\vec{r},t)$$

- Unfortunately we still have an equation with two unknowns, neutron flux and neutron current!

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\underline{\phi(\vec{r},t)} - \nabla \bullet \underline{J(\vec{r},t)}$$

Diffusion Approximation

- If the neutrons have an equal probability of traveling in any direction then it is reasonable to assume that neutrons will "diffuse" from regions of high density (flux) to regions of low density.



$$\phi_1 > \phi_2$$

Diffusion Approximation

- This diffusion process may be expressed mathematically by Fick's Law:

$$\vec{J}\left(\vec{r},t\right) = -D\left(\vec{r},t\right)\nabla\varphi\left(\vec{r},t\right)$$

  – D is an empirical diffusion coefficient

- Fick's law says that the net direction flow of neutrons will always be on the opposite direction of the neutron flux gradient (downhill) with magnitude (rate) determined by the diffusion coefficient D.

Diffusion Approximation

- Using Fick's law to relate the neutron current to the neutron flux gradient is an approximation – the Diffusion Approximation
- Good Approximation:
  - In large, highly scattering materials
- Bad Approximation:
  - Near leakage boundaries
  - Near strong absorbers or sources
  - In a vacuum
  - When scattering is strongly anisotropic

Neutron Diffusion Equation

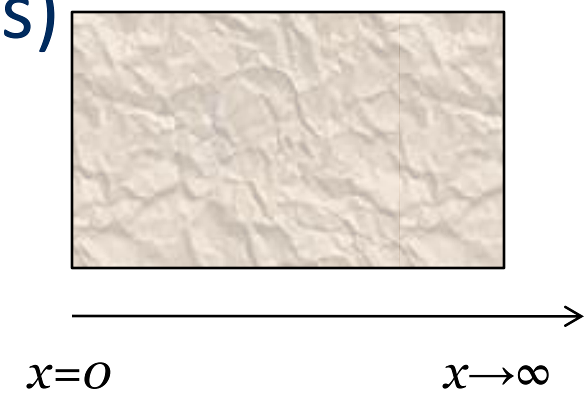
$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\phi(\vec{r},t) + \nabla \bullet D(\vec{r},t)\nabla(\vec{r},t)\phi(\vec{r},t)$$

- We now have a differential equation in one variable (scalar neutron flux)
- Too bad this equation is too complicated to solve analytically.
- If we can't solve the full equation, let's start simplifying until we get something that we can solve!

One Speed, 1-D, Steady-State Neutron Diffusion Equation

- Boundary Conditions (4 types)
  - 1. Finite Flux Requirement

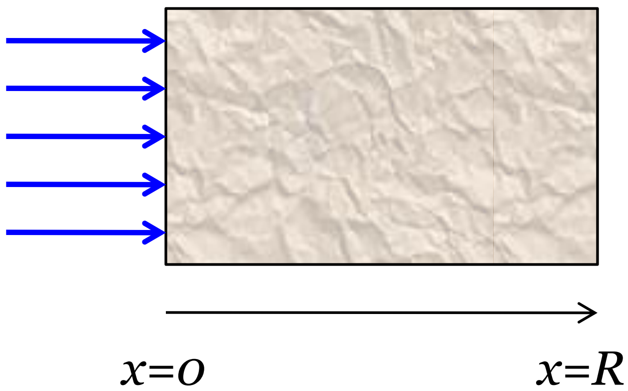
$$\lim_{x \to \infty} \phi(x) < \infty$$



  - 2. Incident Current Condition

$$J(0) = s_0$$

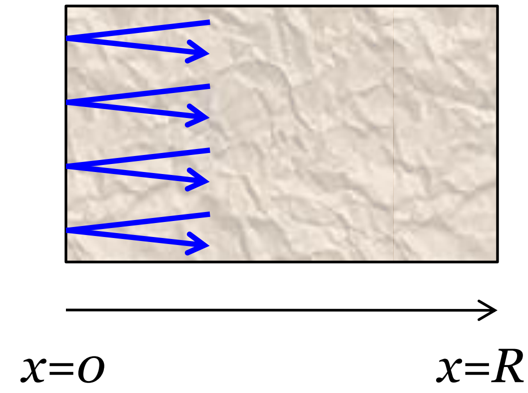
Using Fick's Law: $$-D\frac{d\phi(x)}{dx}\bigg|_{x=0} = s_0$$



One Speed, 1-D, Steady-State Neutron Diffusion Equation

- Boundary Conditions
  - 3. Reflecting Boundary

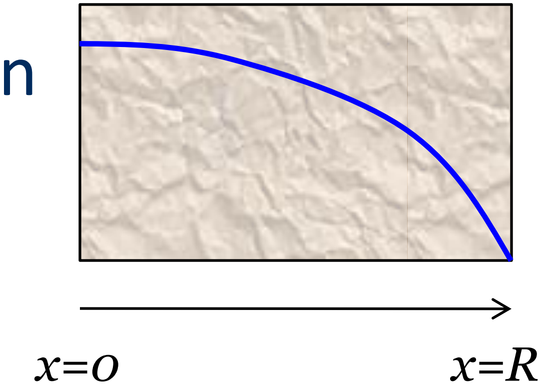
$$J(0)=0$$



Using Fick's Law: $$\left.\frac{d\phi(x)}{dx}\right|_{x=0}=0$$

  - 4. Zero Flux (Escape) Condition

$$\phi(R)=0$$



Diffusion Length

$$-D\frac{d^2}{d^2x}\phi(x)-\Sigma_a\,\phi(x)=s(x)$$

- Rewrite this equation as

$$\frac{d^2}{d^2x}\phi(x)-\frac{1}{L^2}\phi(x)=-\frac{s(x)}{D}$$

- The variable L is called the diffusion length and L^2 is called the diffusion area

$$L=\sqrt{\frac{D}{\Sigma_a}} \qquad L^2=\frac{D}{\Sigma_a}$$

  - The diffusion length has units of cm and the diffusion area of cm^2

Differential Equation Solution

- The differential equation

$$\frac{d^2}{d^2x}\phi(x)-\frac{1}{L^2}\phi(x)=-\frac{s(x)}{D}$$

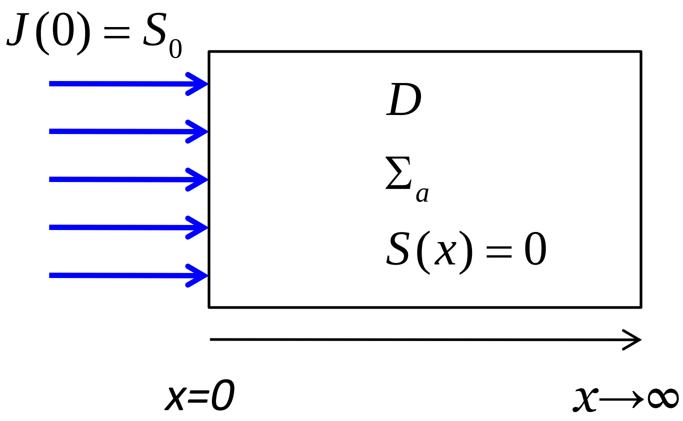
has the following solution

$$\phi(x)=\phi_{\text{homogeneous}}(x)+\phi_{\text{particular}}(x)$$

$$\phi_{\text{homogeneous}}(x)=c_1e^{-x/L}+c_2e^{x/L}$$

where the homogeneous solution must satisfy the boundary conditions and the particular solution our source term

Example 1: Semi-Infinite Slab



- No particular solution in this case

$$\phi(x) = c_1 e^{-x/L} + c_2 e^{x/L}$$

- Boundary Conditions

$$J(0) = S_0$$

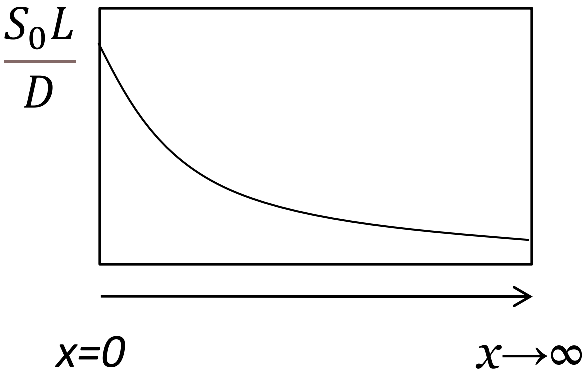
$$\lim_{x \rightarrow \infty} \phi(x) < \infty$$

Example 1: Semi-Infinite Slab

- Finite Flux

$$\lim_{x\to\infty}\left(c_1 e^{-x/L}+c_2 e^{x/L}\right)<\infty$$

$$c_2=0$$



- Incident Current

$$J(0)=-D\left.\frac{d\varphi(x)}{dx}\right|_{x=0}=S_0$$

$$-D\left.\frac{d\varphi(x)}{dx}\right|_{x=0}=D\left.\frac{c_1 e^{-x/L}}{L}\right|_{x=0}=S_0$$

$$c_1=\frac{S_0 L}{D}$$

- Flux Solution

$$\phi(x)=\frac{S_0 L}{D}e^{-x/L}$$

Fission Neutron Source

$$\nu\Sigma_f(\vec{r},t)\,\phi(\vec{r},t)$$

- This is referred to as a multiplying source term (opposed to a fixed source term) because the magnitude of the source at every point depends on the flux at the point.
- Replacing our fixed-source gives us the diffusion equation in a multiplying medium

$$-D\frac{d^2}{d^2x}\phi(x)+\Sigma_a\,\phi(x)=\nu\Sigma_f(\vec{r},t)\,\phi(\vec{r},t)$$

The k Eigenvalue

- To ensure we have a solution for any system configuration we imagine that the number of neutrons emitted per fission can be changed $\nu \rightarrow \frac{\nu}{k}$
- In this way <u>any</u> system can be made critical by choosing the appropriate value of k

$$-D\frac{d^2}{d^2x}\phi(x)+\Sigma_a\,\phi(x)=\frac{\nu}{k}\Sigma_f\,(\vec{r},t)\,\phi(\vec{r},t)$$

Diffusion in Multiplying Media

$$-D\frac{d^2}{dx^2}\phi(x)+\Sigma_a\,\phi(x)=\frac{\nu}{k}\Sigma_f\phi(x)$$

- Define material buckling:

$$B_m{}^2=\frac{\left(\frac{\nu}{k}\Sigma_f-\Sigma_a\right)}{D}$$

- We can then write the one-group reactor problem a $\frac{d^2}{dx^2}\phi(x)+B^2\,\phi(x)=0$

Multiple Solutions

- The subscript *n* indicates there are many solutions which solve the one-group reactor equation, called harmonic modes

$$\phi_n(x) = c_n \cos\left(\frac{n\pi x}{a}\right)$$

- Really, any of the possible solutions are valid solutions

$$\phi(x) = \sum_{n=1,3,5,\ldots}^{\infty} c_n \cos\left(\frac{n\pi x}{a}\right) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{(2n-1)\pi x}{a}\right)$$

Fundamental Mode

- As n increase k decreases

$$k_n = \frac{\nu\Sigma_f}{DB_n^{\ 2} + \Sigma_a} = \frac{\nu\Sigma_f}{\frac{D\pi^2}{a^2}(2n-1)^2 + \Sigma_a}$$

- Higher order modes become increasingly subcritical (decreasing neutron population). If we wait long enough only 1st mode remains, called fundamental mode

$$k_1 = \frac{\nu\Sigma_f}{\frac{D\pi^2}{a^2} + \Sigma_a} = k_{\text{eff}}$$

- Reactor properties determined by the fundamental mode

Criticality Condition

- Set k=1 and solve for geometric buckling

$$k = \frac{\nu\Sigma_f}{DB_g^{\,2} + \Sigma_a} \rightarrow DB_g^{\,2} + \Sigma_a = \nu\Sigma_f$$

$$B_g^{\,2} = \frac{\nu\Sigma_f - \Sigma_a}{D} = B_m^{\,2}$$

- In a critical system the geometric buckling is equal to the material buckling
  - To achieve criticality the system requires compatible materials and geometric configuration

Fundamental Mode

- We now know the flux and multiplication are described by the fundamental mode

$$\phi_1(x) = c_1 \cos\left(\frac{\pi x}{a}\right)$$

but we still need to find c_1

- To find a unique value of c_1 we can write it in terms of the current power level!

Power Calculation

- The power produced in the reactor is

$$P = E_R \Sigma_f \int_{-a/2}^{a/2} \phi(x)dx$$

$$= \frac{aE_R\Sigma_f c_1}{\pi} \sin\left(\frac{\pi x}{a}\right)\Bigg|_{-a/2}^{a/2} = \frac{2aE_R\Sigma_f c_1}{\pi}$$

$$c_1 = \frac{P\pi}{2aE_R\Sigma_f}$$

Power in watts/cm[2]

Macroscopic fission cross section

Slab width

Energy recoverable from fission (joules/fission)

$$\phi(x) = \frac{P\pi}{2aE_R\Sigma_f} \cos\left(\frac{\pi x}{a}\right)$$

General Geometries

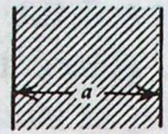
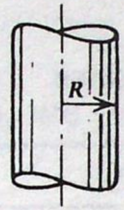
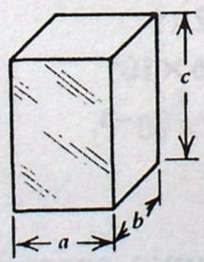
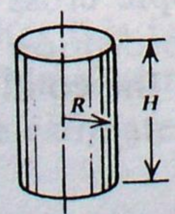
- We have solved the one-group reactor criticality problem for a slab by finding the geometric buckling and equating to the material buckling
- Can we do the same thing for other geometries?
  - Yes, in fact it is the exact same process

$$k = \frac{\nu\Sigma_f}{D{B_g}^2 + \Sigma_a}$$

Process for General Geometries

- Find geometric buckling
  - Solve differential equation for desired geometry OR
  - See Table 6.2 in L&B for common geometries
- Find the constant in terms of total power
- Find k by substituting the geometric buckling equation in
  - If searching for critical dimension then set material buckling to geometric buckling and solve for the desired dimension

TABLE 5-1 Geometric Bucklings and Critical Flux Profiles Characterizing Some Common Core Geometries

		Geometric Buckling B_g^2	Flux profile
Slab		$\left(\frac{\pi}{\tilde{a}}\right)^2$	$\cos\frac{\pi x}{\tilde{a}}$
Infinite Cylinder		$\left(\frac{\nu_0}{\tilde{R}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)$
Sphere		$\left(\frac{\pi}{\tilde{R}}\right)^2$	$r^{-1}\sin\left(\frac{\pi r}{\tilde{R}}\right)$
Rectangular Parallelepiped		$\left(\frac{\pi}{\tilde{a}}\right)^2+\left(\frac{\pi}{\tilde{b}}\right)^2+\left(\frac{\pi}{\tilde{c}}\right)^2$	$\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$
Finite Cylinder		$\left(\frac{\nu_0}{\tilde{R}}\right)^2+\left(\frac{\pi}{\tilde{H}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)\cos\left(\frac{\pi z}{\tilde{H}}\right)$