University of Pittsburgh

# ME/ENGR 2100 Fundamentals of Nuclear Engineering

## Neutron Diffusion:

## Diffusion Theory

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# Related Reading

- Chapter 5 of Duderstadt and Hamilton

OR

- Sections 5.1-5.7 Lamarsh and Baratta
- Sections 6.1-6.4 Lamarsh and Baratta

# Learning Objectives

- Write the neutron diffusion equation describing neutron balances (gains and losses) in non-multiplying systems

- Explain under what conditions the diffusion approximation to the continuity equation is valid and how its use in reactor analysis is justified

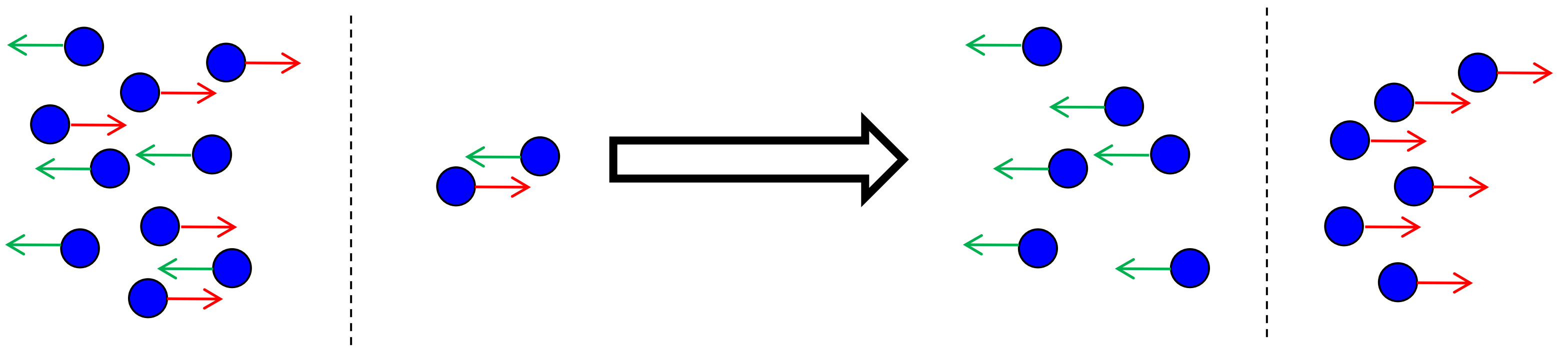
# The equation of continuity

$$\begin{bmatrix}\text{Rate of change of}\\ \text{number of}\\ \text{neutrons in V}\end{bmatrix} = \begin{bmatrix}\text{Rate of}\\ \text{production of}\\ \text{neutrons in V}\end{bmatrix} - \begin{bmatrix}\text{Rate of}\\ \text{absorption of}\\ \text{neutrons in V}\end{bmatrix} - \begin{bmatrix}\text{Rate of}\\ \text{leakage of}\\ \text{neutrons in V}\end{bmatrix}$$

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\underline{\phi(\vec{r},t)} - \nabla \bullet \underline{J(\vec{r},t)}$$

# Diffusion Approximation

- If the neutrons have an equal probability of traveling in any direction then it is reasonable to assume that neutrons will "diffuse" from regions of high density (flux) to regions of low density.



$$\phi_1 > \phi_2$$

# Diffusion Approximation

- This diffusion process may be expressed mathematically by Fick's Law:

$$\vec{J}\left(\vec{r},t\right)=-D\left(\vec{r},t\right)\nabla\varphi\left(\vec{r},t\right)$$

  - $D$ is an empirical diffusion coefficient

- Fick's law says that the net direction flow of neutrons will always be on the opposite direction of the neutron flux gradient (downhill) with magnitude (rate) determined by the diffusion coefficient $D$.

# Diffusion Approximation

- Using Fick's law to relate the neutron current to the neutron flux gradient is an approximation – the Diffusion Approximation
- Good Approximation:
  - In large, highly scattering materials
- Bad Approximation:
  - Near leakage boundaries
  - Near strong absorbers or sources
  - In a vacuum
  - When scattering is strongly anisotropic

# Neutron Diffusion Equation

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\phi(\vec{r},t) + \nabla\bullet D(\vec{r},t)\nabla(\vec{r},t)\phi(\vec{r},t)$$

- We now have a differential equation in one variable (scalar neutron flux)
- Too bad this equation is too complicated to solve analytically.
- If we can't solve the full equation, let's start simplifying until we get something that we can solve!

# Neutron Diffusion Equation

- Derived from the continuity equation but with an explicit approximation made to the leakage term
  - Fick's Law used to describe diffusion of gas molecules in air
  - Laplacian operator common to heat equation as well
- Approximation works well in regions where neutron scattering is the dominant process
  - Such as nuclear reactors!
  - What does this approximation mean to nuclear shielders?
  - Can a neutron beam be accurately described by the diffusion equation?

# Neutron Diffusion Equation

$$\cancelto{0}{\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t}} = s(\vec{r},t) - \Sigma_a(\vec{r},t)\phi(\vec{r},t) + \nabla \bullet D(\vec{r},t)\nabla(\vec{r},t)\phi(\vec{r},t)$$

- If we can't solve the full equation, let's start simplifying until we get something that we can solve!
- Simplification #1
  - **Steady State**: $$\frac{d}{dt}\left[\varphi\left(\vec{r},t\right)\right] = 0$$

# Steady-State Neutron Diffusion Equation

- Rearrange: source terms on right, removal on left.
- Drop the *t* variables, we don't need them.

$$-\nabla \bullet D(\vec{r})\nabla(\vec{r})\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = s(\vec{r})$$

- Better… but still pretty complicated.
- We still have complicated operators over the spatial (divergence) variable

# Steady-State Neutron Diffusion Equation

$$-\nabla\bullet D(\cancel{\vec{r}})\nabla(\vec{r})\phi(\vec{r})+\Sigma_a(\cancel{\vec{r}})\phi(\vec{r})=s(\vec{r})$$

- <u>Simplification 2</u>
  - **Homogeneous material**
  - Cross sections and diffusion coefficient are independent of position.

# Steady-State Neutron Diffusion Equation

Note:

$$\nabla \cdot \nabla f(x) = \nabla^2 f(x)$$

(Laplacian operator):

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = s(\vec{r})$$

- <u>Simplification 2</u>
  - Based on the homogeneous material assumption we can eliminate the dependence of cross sections (and $D$) on $\vec{r}$.
  - This allows $D$ to be factored outside of the divergence operator.

# Laplacian in Different Coordinate Systems

Cartesian $$\nabla^2\phi(x,y,z)=\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2}{\partial y^2}\phi+\frac{\partial^2}{\partial z^2}\phi$$

Cylindrical $$\nabla^2\phi(\rho,\theta,z)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\phi\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\phi+\frac{\partial^2}{\partial z^2}\phi$$

Spherical $$\nabla^2\phi(r,\theta,\gamma)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\phi\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\phi\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial^2\gamma}\phi$$

# 1-D Steady-State Neutron Diffusion Equation

$$-D\overset{\frac{d^2}{d^2x}}{\nabla^2}\phi(\overset{x}{\vec{r}})+\Sigma_a\phi(\overset{x}{\vec{r}})=s(\overset{x}{\vec{r}})$$

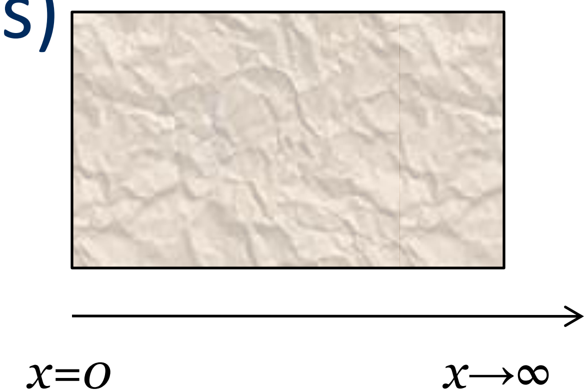
$$\boxed{-D\frac{d^2}{d^2x}\phi(x)+\Sigma_a\,\phi(x)=s(x)}$$

- Simplification 3
  - **(1-D) One spatial dimension** $(x)$
  - Replace $\vec{r}$ by $x$
  - Laplacian $\nabla^2$ becomes ordinary derivative $\dfrac{d^2}{d^2x}$

# One Speed, 1-D, Steady-State Neutron Diffusion Equation

- Boundary Conditions (4 types)
  - 1. Finite Flux Requirement

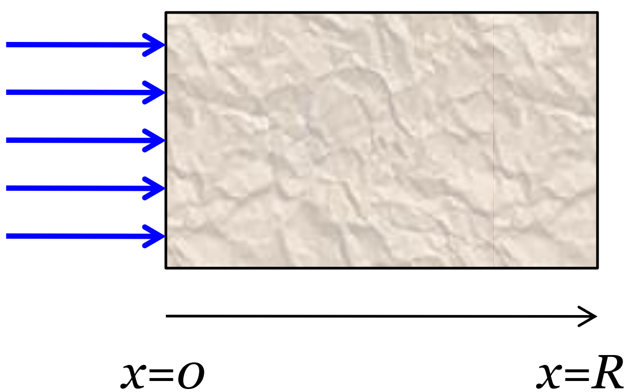
$$\lim_{x \to \infty} \phi(x) < \infty$$




  - 2. Incident Current Condition

$$J(0) = s_0$$

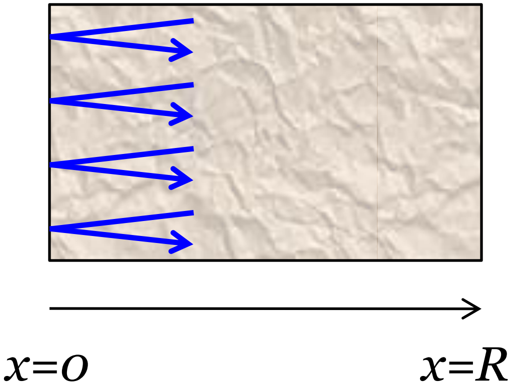
Using Fick's Law: $$-D\left.\frac{d\phi(x)}{dx}\right|_{x=0} = s_0$$



# One Speed, 1-D, Steady-State Neutron Diffusion Equation

- Boundary Conditions
  - 3. Reflecting Boundary

$$J(0)=0$$




Using Fick's Law: $$\left.\frac{d\phi(x)}{dx}\right|_{x=0}=0$$

  - 4. Zero Flux (Escape) Condition

$$\phi(R)=0$$